Trembling-Hand Perfect Nash Equilibrium

intuition: a good property for a solution concept is that players still want to play their strategies even if there's a small probability that the other player makes a mistake

\[
\begin{array}{cc}
  L & R \\
  t & (8, 4) & (3, 3) \\
  b & (5, 0) & (3, 2) \\
\end{array}
\]

there are 2 Nash equilibria to this game, \((t, L)\) and \((b, R)\): the second Nash pair of strategies is a Nash equilibrium, even though it involves player #1 playing a weakly dominated strategy

strategy \(b\) makes sense (sort of) for player #1 only if she is absolutely, 100% certain that player #2 is going to play \(R\) and not \(L\)
trembling-hand perfectness is a “refinement” of Nash equilibrium: a further restriction on the properties of Nash equilibrium which rules out “implausible” outcomes such as \((b, R)\) in the game on the previous page

loosely: a pair of pure strategies \((s, t)\) for players #1 and #2 is a trembling-hand perfect Nash equilibrium if and only if \(s\) is a best reaction for player #1 not only to the pure strategy \(t\) of player #2, but also to “trembles” by player #2: mixed strategies in which player #2 plays each of his pure strategies with some very small but positive probability, and \(t\) with a probability close to, but less than, 1 [with analogous conditions on \(t\) being his best response to trembles from \(s\) by player #1]
more formally

the most useful formal definition is probably Proposition 8.F.1 of *Mas–Colell, Whinston and Green*, which takes their original definition and makes it more useful

a strategy $\tau_i$ for player $i$ is **completely mixed** if it puts some strictly positive weight on each of player $i$’s pure strategy

A profile of mixed strategies $(\sigma_1, \sigma_2, \ldots, \sigma_I)$ for players 1, 2, $\ldots$, $I$ is a trembling–hand perfect Nash equilibrium if there is some sequence of completely mixed strategies $\tau^k = (\tau^k_1, \tau^k_2, \ldots, \tau^k_I)$, with $k = 1, 2, \ldots, \infty$ for the players such that

(i) $\tau^k_i \to \sigma_i$ for each player $i$

(ii) $\sigma_i$ is the best response for player $i$ to each $(\tau^k_1, \tau^k_2, \ldots, \tau^k_{i-1}, \tau^k_{i+1}, \ldots, \tau^k_I)$
if $\sigma$ is a trembling–hand perfect Nash equilibrium, then strategy $\sigma_i$ cannot be a weakly dominated strategy for player $i$.

with only two players, that’s it: the set of trembling–hand perfect Nash equilibria to any game is exactly the set of Nash equilibria in which no player is playing a weakly dominated strategy.

but with more than two players, trembling–hand perfectness may get rid of more equilibria: when $I > 2$, there may be Nash equilibria which are not trembling–hand perfect, even though no player is playing a weakly dominated strategy.

nice property: there always is at least one trembling–hand perfect Nash equilibrium to any game in strategic form.
not–so–nice property

with 2 players, a pair of strategies may be a trembling–hand perfect Nash equilibrium, even if one of the strategies can be removed during iterated deletion of weakly dominated strategies

\((b, R)\) is a trembling–hand perfect Nash equilibrium to

\[
\begin{array}{ccc}
  & L & C & R \\
 t & (8, 4) & (4, 0) & (3, 3) \\
 b & (5, 0) & (6, 1) & (3, 2) \\
\end{array}
\]

even though

\[
\begin{array}{ccc}
  & L & C & R \\
 t & (8, 4) & - & (3, 3) \\
 b & (5, 0) & - & (3, 2) \\
\end{array}
\]