GS/ECON 5300 Answers to Assignment 1 January 2006

*Q*1. A country contains only two types of people, rich and poor. The gross income of each type is fixed. However, the number of rich people is not fixed : it is a decreasing function of the net–of–tax income of rich people (as rich people emigrate in response to lower incomes). If taxes collected from the rich are given to the poor, what tax rate on rich people's income maximizes the welfare of poor people?

*A*1. Let y be the gross income of the rich, and T the total taxes collected from the rich. Then the number of rich people is some function $N(y-T)$, with $N' < 0$.

The total money collected from taxing the rich is thus

$$TR = TN(y-T)$$

The poor people want to maximize TR, since the tax revenue will be distributed to them. So they choose T to maximize $TN(y-T)$, yielding a first–order condition

$$N(y-T) + TN'(y-T) = 0$$

If η denotes the (absolute value of the) elasticity of rich population with respect to their net income

$$\eta \equiv -N'(y-T)\frac{y-T}{N}$$

then the the first–order condition can be written

$$\frac{T}{y-T} = \frac{1}{\eta}$$

The taxes collected from the rich, as a fraction of their after–tax income, should equal the elasticity of their numbers with respect to their income. That is, the more mobile they are (that is, the more responsive they are to income changes), the less they will be taxed.

*Q*2. Suppose that a person lives for 2 periods, 0 and 1, receiving the exogenous certain income stream (Y_0, Y_1) in each of the two periods. If her preferences over present and future consumption can be represented by the utility function

$$U(C_0, C_1) = -\frac{1}{C_0} - \frac{1}{C_1}$$

which of the following is true? — (*a*) her saving must be an increasing function of its net rate of return (*b*) her saving must be a decreasing function of its net rate of return (*c*) her saving could increase or decrease with its net rate of return — Explain your answer.

*A*2. This person has constant elasticity of substitution preferences. Her utility function can be written

$$U(C_0, C_1) = [C_0^{\rho} + C_1^{\rho}]^{1/\rho}$$

with $\rho = -1$.

That means that her demand function for consumption in period 0 can be written

$$C_0 = \frac{Y}{1 + p^{1/2}}$$

using example 1.1 of Jehle and Reny, for example, where p is the price of period–1 consumption, and Y is the total value of her endowment, and where the price of period–0 consumption has been set equal to 0.

That means that

$$Y = Y_0 + pY_1$$

so that her demand for current consumption can be written

$$C_0 = \frac{Y_0}{1 + \sqrt{p}} + \frac{pY_1}{1 + \sqrt{p}} \qquad (2 - 1)$$

Now the first term on the right side of equation $(2 - 1)$ is a **decreasing** function of p, while the second term is an **increasing** function of p. That means that if Y_1 is very small, relative to Y_1, C_0 will decrease with p, while if Y_1 is very large relative to Y_1, C_0 will increase with p.

But saving is just $Y_0 - C_0$, so saving increases with p whenever C_0 decreases with p. And p is the price of future consumption

$$p = \frac{1}{1 + i}$$

if i is the net rate of return to saving.

So the answer is c : if most of her income is earned in period 0, the income effect dominates, and her saving will decrease with the rate of return, and if most of her income is earned in period 1 then the substitution effect dominates and her saving will increase with the rate of return.

*Q*3. What is the elasticity of current saving, with respect to a permanent change in the tax rate on the return to saving, of someone who works for T_1 years, and then is retired for T_2 years, if her (exogenous, certain) income in year t is

$$y_t = y_0 e^{\gamma t}$$

and if her lifetime utility is defined by

$$U = \int_0^{T_1 + T_2} e^{-\rho t} \ln c_t dt$$

where c_t is her consumption rate t periods after she begins her working life, where γ and ρ are both positive parameters, and where the net return to saving $(1-\tau)r$ is greater than ρ (where τ is the tax rate on the return to saving, and r the gross interest rate), and less than γ?

*A*3. The consumer's problem is to maximize $\int_0^{T_1+T_2} e^{-\rho t} \ln c_t dt$ subject to her lifetime budget constraint

$$\int_0^{T_1+T_2} c_t e^{-it} dt \leq y_0 \int_0^{T_1} e^{(\gamma-i)t} dt \qquad (3-1)$$

where

$$i = r(1-t)$$

is her net return to saving. The constraint $(3-1)$ is just the requirement that the present value of her consumption not exceed the present value of her lifetime earnings.

The above maximization can be accomplished by setting up the Lagrangean

$$\mathcal{L} = \int_0^{T_1+T_2} [e^{-\rho t} \ln c_t - \lambda c_t e^{-it}] dt + \lambda y_0 \int_0^{T_1} e^{(\gamma-i)t} dt \qquad (3-2)$$

where λ is the multiplier associated with the lifetime budget constraint $(3-1)$.

Maximizing the Lagrangean with respect to each instantaneous consumption level c_t yields the first–order condition

$$e^{-\rho t}\frac{1}{c_t} - \lambda e^{-it} = 0 \qquad (3-3)$$

which holds for each point in time t. Equation $(3-3)$ can be written

$$c_t = \frac{1}{\lambda} e^{(i-\rho)t} \qquad (3-4)$$

Differentiating equation $(3-4)$ with respect to time, and recognizing that the multiplier λ is a constant yields

$$\dot{c}_t = \frac{1}{\lambda}(i-\rho)e^{(i-\rho)t} \qquad (3-5)$$

so that

$$\frac{\dot{c}_t}{c_t} = i - \rho \qquad (3-6)$$

Equation $(3-6)$ says that the person's optimal consumption grows at the rate $i-\rho$, so that her consumption at time t is

$$c_t = c_0 e^{(i-\rho)t} \qquad (3-7)$$

where c_0 is her initial consumption level.

Substitution of $(3-7)$ into the budget constraint $(3-1)$ implies

$$c_0 \int_0^{T_1+T_2} e^{(i-\rho)t} dt = y_0 \int_0^{T_1} e^{(\gamma-i)t} dt \qquad (3-8)$$

so that

$$c_0 = \frac{i-\rho}{\gamma - i} \frac{e^{(\gamma-i)T_1} - 1}{e^{(i-\rho)(T_1+T_2)} - 1} y_0 \tag{3 – 9}$$

Since initial saving is $y_0 - c_0$, the derivative of saving with respect to the tax rate on the return to saving is r times the derivative of the right side of $(3-9)$ with respect to i. [That is : $\partial s_0/\partial i = -\partial c_0/\partial i$, and $\partial c_0/\partial \tau = -r\partial c_0/\partial i$.]

Differentiating equation $(3-9)$ with respect to i

$$\frac{\partial c_0}{\partial i} = c_0[\frac{1}{i-\rho} + \frac{1}{\gamma - i} - \frac{T_1 e^{(\gamma-i)T_1}}{e^{(\gamma-i)T_1} - 1} - \frac{(T_1+T_2)e^{(i-\rho)(T_1+T_2)}}{e^{(i-\rho)(T_1+T_2)} - 1}] \tag{3 – 10}$$

If the expression in square brackets in equation $(3-10)$ is denoted $-A$, then

$$\frac{\partial s}{\partial \tau}\frac{\tau}{s} = -(A)\frac{c_0}{s}\frac{\tau}{1-\tau} i \tag{3 – 11}$$

The expression A must be positive. The overall elasticity can be fairly large, if, for example, γ and T_1 are fairly large : an elasticity of saving with respect to the tax rate of 1 is not implausible for someone with 40 years to go in her working life, if her earnings are expected to grow at 6% per year, and if the current tax rate on saving is 40%.

*Q*4. How would an individual's investment in a risky asset vary with the tax rate t on the net return to investment if (i) her utility–of–wealth function was $u(W) = \ln W$; (ii) she had a fixed initial wealth to divide between a risky asset and a safe asset ; (iii) the safe asset earned a sure net rate of return r_0 ; (iv) the risky asset earned a return $r_g > r_0$ with probability π, and a return $r_b < r_0$ with probability $1-\pi$?

*A*4. The person's first–order condition for her optimal choice portfolio choice can be written

$$E[U'(W)(r - r_0)] = 0 \tag{4 – 1}$$

which here implies that

$$\frac{\pi(r_g - r_0)}{1 + r_0(1-t) + x(r_g - r_0)(1-t)} = \frac{(1-\pi)(r_0 - r_b)}{1 + r_0(1-t) - x(r_0 - r_b)(1-t)} \tag{4 – 2}$$

where x is the fraction of her wealth which is invested in the risky asset.

Because the utility–of–wealth function exhibits constant relative risk aversion, the fraction of wealth invested in the risky asset does not depend on the person's initial wealth.

Equation $(4-2)$ can be solved for x :

$$x = \frac{[1 + r_0(1-t)][Er - r_0]}{(r_g - r_0)(r_0 - r_b)(1-t)} \tag{4 – 3}$$

where Er is the expected return on the risky asset

$$Er \equiv \pi r_g + (1-\pi) r_b$$

Equation $(4-3)$ shows that the person's investment in the risky assset must **increase** with the tax rate t, since it can be written

$$x = [\frac{1}{(1-t)} + r_0]\frac{(Er - r_0)}{(r_g - r_0)(r_0 - r_b)} \qquad (4-4)$$

In particular,

$$\frac{\partial x}{\partial t} = \frac{1}{1 + r_0(1-t)}x$$

*Q*5. Re–do question #4 above if the return r_b in the bad state were negative, and if the tax authorities did not allow any deductions from tax for investment losses.

*A*5. Now the first–order condition for the person's share x of the portfolio invested in the risky asset is

$$\pi\frac{(1-t)(r_g - r_0)}{1 + r_0(1-t) + x(1-t)(r_g - r_0)} = (1-\pi)\frac{(1-t)r_0 - r_b}{1 + r_0(1-t) - x(r_0(1-t) - r_b)} \qquad (5-1)$$

so that

$$x = \frac{[1 + r_0(1-t)][\pi(1-t)r_g + (1-\pi)r_b - (1-t)r_0]}{(1-t)[(1-t)r_0 - r_b](r_g - r_0)} \qquad (5-2)$$

Now it is not the case that x must increase with the tax rate t. If t is large enough, the person would decide not to hold **any** of the risky asset. Equation $(5-2)$ implies that $x = 0$ when

$$t = 1 - \frac{1-\pi}{\pi}\frac{r_0 - r_b}{r_g - r_0} < 1$$

But it is also not true that x **must** decline with the tax rate here. For example, if $\pi = 0.89$, $r_0 = 0.4$, $r_g = 0.5$ and $r_b = -0.1$, then an increase in the tax rate on the net return to successful investment from 0 to 1 % would increase the fraction x that the person chose to invest in the risky asset (from 95.2% of her wealth, to 95.38%).