GS/ECON 5300 Answers to Assignment 2 November 2007

*Q*1. Find all the Pareto optimal allocations in the following 2–person, 2–good economy, in which good X is a pure private good and good Z is a pure public good.

The production possibility frontier for the economy has the equation

$$2X + Z = 270$$

where X and Z are the aggregate quantities produced of the pure private good and of the pure public good, respectively.

There are 2 people : person 1's preferences can be represented by the utility function

$$U^1(x_1, z_1) = 2 \ln x_1 + \ln z_1$$

and person 2's by the utility function

$$U^2(x_2, z_2) = \ln x_2 + 2 \ln z_2$$

where x_h and z_h are person h's consumption of the pure private good and pure public good respectively.

*A*1. The marginal rate of transformation here equals 1/2; each increase of 1 unit in Z lowers Z by 1/2.

The Samuelson condition for an efficient allocation is that

$$MRS^1 + MRS^2 = MRT$$

Here

$$MRS^1 = \frac{U_z^1}{U_x^1} = \frac{1}{2}\frac{x_1}{z_1}$$

$$MRS^2 = \frac{U_z^2}{U_x^2} = 2\frac{x_2}{z_2}$$

In an efficient allocation, both people should get to consume the full quantity available of the non–rivalrous public good, so that $z_1 = z_2 = Z$, which means that the Samuelson condition can be written

$$\frac{1}{2}\frac{x_1}{Z} + 2\frac{x_2}{Z} = \frac{1}{2}$$

or

$$x_1 + 4x_2 = Z \qquad (1-1)$$

Optimality also requires that the allocation be on the production possibility frontier, or

$$2x_1 + 2x_2 + Z = 270 \qquad (1-2)$$

Any allocation (x_1, x_2, Z) satisfying equations $(1-1)$ and $(1-2)$, with (x_1, x_2, Z) all non–negative, will be a Pareto optimal allocation.

For example, equations $(1-1)$ and $(1-2)$ can be solved for x_1 and x_2 as functions of x_1 :

$$x_2 = 45 - \frac{1}{2}x_1 \tag{1 – 3}$$

$$Z = 180 - x_1 \tag{1 – 4}$$

So take any x_1 between 0 and 90, and $(x_1, x_2, Z) = (x_1, 45 - \frac{1}{2}x_1, 180 - x_1)$ will be a Pareto optimal allocation.

*Q*2. Suppose that there are only two people in a country. Person 1's preferences can be represented by the utility function

$$U^1(x_1, z_1) = x_1 + Az_1 - \frac{1}{2}\alpha(z_1)^2$$

and person 2's by the utility function

$$U^2(x_2, z_2) = x_2 + Az_2 - \frac{1}{2}\beta(z_2)^2$$

where x_h and z_h are person h's consumption of a pure private good and of a pure public good respectively.

The government knows A, but asks individuals 1 and 2 each to report a value a (for person 1) and b (for person 2) to measure the slope parameter in their demand curves for the public good.

The government promises to choose a public good level

$$Z^* = \frac{2A - c}{a + b}$$

and to tax person h

$$\frac{cZ^*}{2} + T_h$$

where

$$T_1 = (Z^* - Z_1)[\frac{b}{2}(Z^* + Z_1) - \frac{2A - c}{2}]$$

$$T_2 = (Z^* - Z_2)[\frac{a}{2}(Z^* + Z_2) - \frac{2A - c}{2}]$$

where the variables Z_1 and Z_2 are defined by

$$Z_1 = \frac{2A - c}{2b}$$

$$Z_2 = \frac{2A - c}{2a}$$

If the government commits to these rules for public good provision and taxation, what values a and b will the people report? Explain your answer.

*A*2. Person 1's (true) utility will be

$$y_1 - (\frac{cZ^*}{2} + T_1) + AZ^* - \frac{1}{2}\alpha(Z^*)^2 \qquad (2-1)$$

if the level of public goods Z^* is provided, and if she must pay a "pivot tax" T_1. She should choose her announced parameter a to maximize expression $(2-1)$, so that

$$[A - \alpha Z^* - \frac{c}{2}]\frac{\partial Z^*}{\partial a} - \frac{\partial T_1}{\partial a} = 0 \qquad (2-2)$$

Here Z_1 does not depend on a, and

$$\frac{\partial T_1}{\partial a} = [\frac{b}{2}(Z^* + Z_1) - \frac{2A - c}{2}]\frac{\partial Z^*}{\partial a} + (Z^* - Z_1)\frac{b}{2}\frac{\partial Z^*}{\partial a} \qquad (2-3)$$

so that $(2-2)$ becomes

$$A - \alpha Z^* - \frac{c}{2} - bZ^* + \frac{2A - c}{2} = 0 \qquad (2-4)$$

or

$$(\alpha + b)Z^* = 2A - c \qquad (2-5)$$

Since $Z^* = \frac{2A-c}{a+b}$, equation $(2-5)$ implies that

$$\frac{\alpha + b}{a + b} = 1 \qquad (2-6)$$

or $a = \alpha$. Person 1 should reveal truthfully her public good demand slope parameter α. Similarly, person 2 should reveal truthfully her public good slope demand parameter β.

[Alternatively : the mechanism described is just the general "partial equilibrium" preference revelation mechanism discussed in class, for the special case in which $f_1(Z) = AZ - \frac{1}{2}\alpha Z^2$ and $f_2(Z) = AZ - \frac{1}{2}\beta Z^2$.]

*Q*3. How much of a deficit or surplus will the government run, if it uses the public good provision and taxation rules defined in question #2 above?

*A*3. Here

$$Z_1 = \frac{a + b}{2b}Z^*$$

so that

$$Z^* - Z_1 = \frac{b - a}{2b}Z^*$$

$$Z^* + Z_1 = \frac{3b + a}{a + b}Z^*$$

therefore

$$T_1 = \frac{(Z^*)^2}{8b}(b - a)^2$$

Similarly

$$T_2 = \frac{(Z^*)^2}{8a}(a-b)^2$$

In equilibrium, people reveal truthfully their preferences, so that $a = \alpha$, $b = \beta$, and

$$Z^* = \frac{2A-c}{\alpha+\beta}$$

So the total surplus collected is

$$T_1 + T_2 = \frac{1}{8}\frac{(2A-c)^2}{\alpha\beta(\alpha+\beta)}(\beta-\alpha)^2 \qquad (3-2)$$

Here the surplus must be non–negative, will be zero only if the two people have exactly the same preferences, and increases with the relative difference in their preferences.

*Q*4. A country consists of 3 million people, 1 million of each type. Each person's preferences can be represented by the utility function

$$U^h = x_h + \ln z_h$$

where x_h is her consumption of a pure private good and z_h her consumption of a pure public good.

The public good must be financed by a proportional income tax, at the rate τ.

Each person's income is exogenous, but different types have different income. Type 1 people have income of 5, type 2 people have income of 10, and type 3 people have income of 30.

The cost of 1 unit of the public good is 3000000.

If people vote over the tax rate τ (with all tax revenue being used for public good provision), which tax rate will be chosen?

*A*4. The average income in the country is 15. So an income tax rate of τ will collect 15τ per person, for a total revenue of 45000000τ. Since each unit of the public good costs 3000000, a tax rate of τ can finance 15τ units of the public good.

So, if the tax rate were τ, a person of income y would have utility of

$$(1-\tau)y + \ln 15\tau$$

Maximizing this expression with respect to τ implies that

$$\frac{15}{\tau} - y = 0$$

or

$$\tau = \frac{1}{y}$$

The higher a person's income, the lower the tax rate she prefers. (The reason? Since preferences are quasi–linear, demand for the public good has an income elasticity of 0. So high–income people have the same demand curve for the public good as low–income people, but pay a higher price for the public good since they bear more of the tax burden.)

Since a person's preferred tax rate is a monotonic function of her income, the median voters here are the people of median income, those with $y = 10$, so that the tax rate which is a Condorcet winner when people vote is $\tau = 0.10$.

*Q*5. If it were also possible to make lump–sum transfers among people, find an allocation which is Pareto–preferred to the one chosen in the voting in question #4 above.

*A*5. In the voting equilibrium, $\tau = 0.1$ and $Z = 15\tau = 1.5$, so the allocation is $(x_1, x_2, x_3, Z) = (4.5, 9, 27, 1.5)$.

Each person's MRS here is $U_z/U_x = 1/Z = 2/3$, so that

$$\sum MRS = 2000000 < MRT = 3000000$$

meaning that "too much" of the public good is provided.

For example. lowering the public good level to 1, and paying each person the same share of the cost saving, which is 0.5, would raise each person's utility by

$$0.5 - \ln 1.5 + \ln 1$$

Since $\ln 1.5 \approx 0.4055$, and $\ln 1 = 0$, the change would make everyone better off.

But note that the Pareto–improvement required side payments. If the public good provsion were lowered simply by reducing the income tax rate, the lower–income people would not be made better off.