

Q1. Find all the Pareto optimal allocations in the following 2-person, 2-input, 2-good economy.

The quantity F produced of food is

$$F = k_F + l_F$$

where k_F and l_F are the quantities of machinery and labour used in food production.

The quantity C of clothing produced is

$$C = \sqrt{k_C l_C}$$

where k_C and l_C are the quantities of machinery and labour used in clothing production.

The total quantities of inputs available in the economy are 120 units of machinery and 240 units of labour. Both of these quantities are fixed.

Person 1's preferences can be represented by the utility function

$$u^1(F_1, C_1) = \ln F_1 + 2 \ln C_1$$

and person 2's by the utility function

$$u^2(F_2, C_2) = 2 \ln F_2 + \ln C_2$$

where (F_1, C_1) is person 1's consumption bundle of food and clothing, and (F_2, C_2) is person 2's.

A1. It is easiest to start with the production side of the economy, to find the allocations of labour and machinery which are efficient in production.

Since labour and machinery are perfect substitutes in the production of food, $MP_F^L = MP_F^K = 1$. The marginal rate of substitution between labour and machinery in food production is 1, regardless of how much (or how little) labour and machinery are being used; the isoquants between labour and machinery are straight lines in the food industry.

In the clothing industry, the production function is $C = \sqrt{k_C l_C}$, so that

$$MP_C^K = \frac{1}{2} \sqrt{\frac{l_c}{k_c}} \quad (1-1)$$

$$MP_C^L = \frac{1}{2} \sqrt{\frac{k_c}{l_c}} \quad (1-2)$$

from partial differentiation of the production function for clothing.

The MRTS in clothing production is the ratio of the marginal products of the two inputs, so that (1-1) and (1-2) imply that

$$MRTS_C^{KL} = \frac{MP_C^K}{MP_C^L} = \frac{l_c}{k_c} \quad (1-3)$$

Efficiency in production requires the marginal rates of technical substitution to be the same in both food and clothing industries, so that $MRTS_F^{KL} = MRTS_C^{KL}$. Since $MRTS_F^{KL} = 1$, regardless of what are l_F and k_F , equation (1 – 3) implies that for efficiency in production, we must have

$$k_C = l_C \quad (1 - 4)$$

So any allocation (k_F, l_F, k_C, l_C) will be efficient in production if $k_C = l_C$, $k_F = 120 - k_C$ and $l_F = 240 - l_C$. Allocations such as $(k_F, l_F, k_C, l_C) = (60, 180, 60, 60)$, $(k_F, l_F, k_C, l_C) = (100, 220, 20, 20)$, and $(k_F, l_F, k_C, l_C) = (10, 130, 110, 110)$ are all efficient in production.

Now that the efficient (in production) allocation of inputs to the two industries has been derived, the production possibility frontier can be found. Take an allocation in which z units of machinery are allocated to the clothing industry. If it is efficient in production, it must be true that $l_C = z$ as well. In this allocation $C = \sqrt{k_C l_C} = \sqrt{z \cdot z} = z$. If $k_C = l_C = z$, then $k_F = 120 - z$ and $l_F = 240 - z$, so that

$$F = k_F + l_F = (120 - z) + (240 - z) = 360 - 2z \quad (1 - 5)$$

Every increase of z by 1 unit will increase C by 1 unit, and decrease F by 2 units. So the marginal rate of transformation is 2 : increasing clothing production by 1 unit reduces food production by 2 units.

The production possibility here is a straight line, since each move of 1 unit of capital and of labour from food to clothing production increases C by 1 and reduces F by 2. The equation of that straight line is

$$F = 360 - 2C \quad (1 - 6)$$

This result can also be seen from calculating the MRT as the ratios of the marginal products ; when $k_C = l_C$ (as must be true if the allocation is efficient in production), $MP_C^K = \frac{1}{2}$, and $MP_F^K = 1$.

However : the above analysis applies only when (k_F, l_F, k_C, l_C) are all positive. Once C has reached 120, all the capital is employed in the food industry. [At that point $(k_F, l_F, k_C, l_C) = (0, 120, 120, 120)$ and $C = 120, F = 120$.] Further increases in clothing production require movement of labour from the food industry to the clothing industry. So, for example, the allocation $(k_F, l_F, k_C, l_C) = (0, 100, 120, 140)$ is also efficient in production ; it represents an allocation on the bottom edge of the Edgeworth box (if capital is measured along the vertical, and allocations to food are measured from the bottom left of the box).

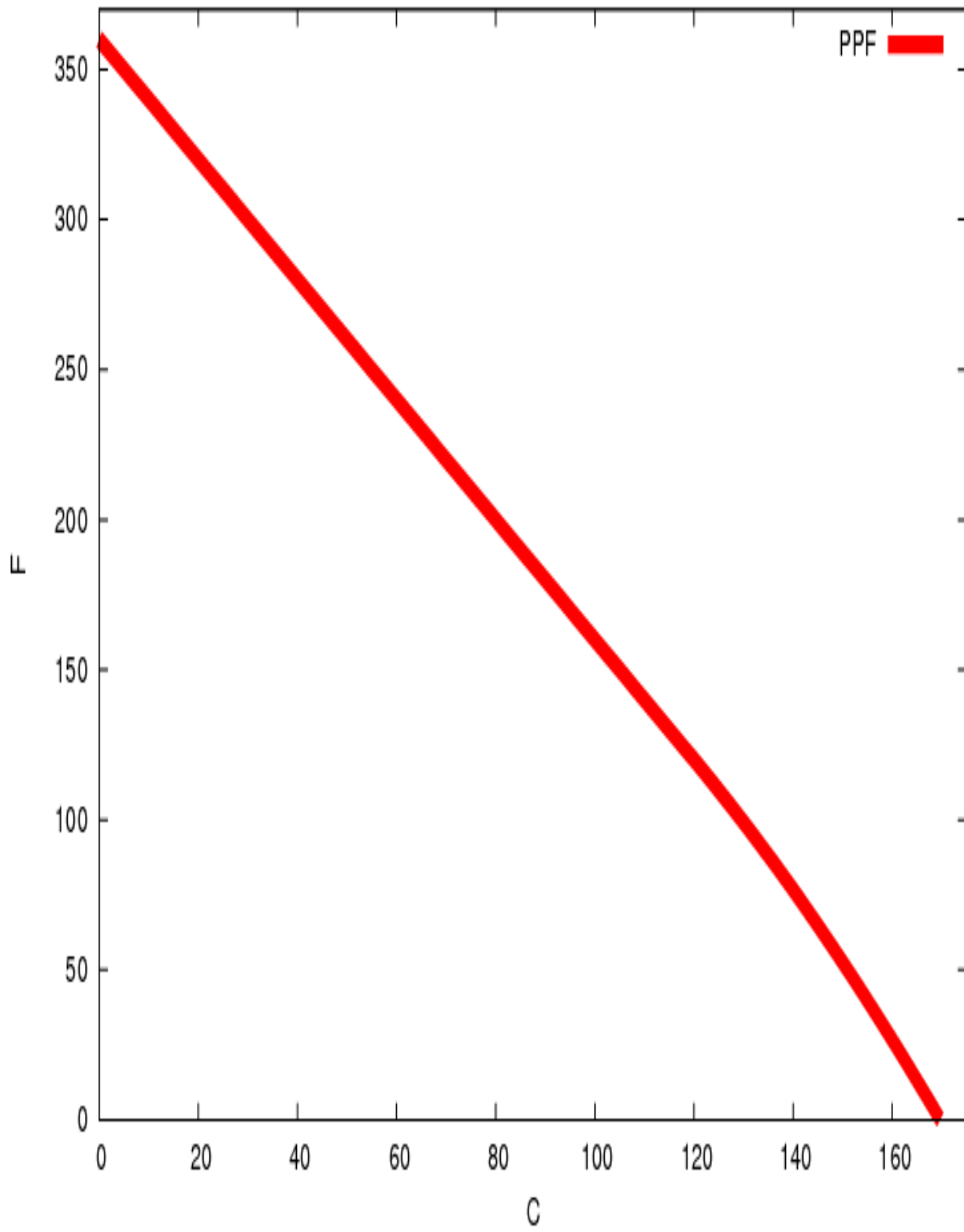
If $(k_F, l_F, k_C, l_C) = (0, 240 - Z, 120, Z)$, with $120 \leq Z \leq 240$, then

$$F = 240 - Z \quad ; \quad C = \sqrt{120Z}$$

so that

$$\frac{\partial F}{\partial Z} = -1 \quad ; \quad \frac{\partial C}{\partial Z} = \sqrt{\frac{30}{Z}}$$

Question 1 : the Production Possibility Frontier



and the PPF has a slope of $\sqrt{\frac{Z}{30}}$, which increases from 2 (at $Z = 120$) to $2\sqrt{2}$ (at $Z = 240$).

Now to the consumption side. For efficiency in exchange, the 2 people must have the same MRS between food and clothing. And for overall efficiency, that common MRS must equal the MRT, which has already been shown to equal 2 – at least when $C \leq 120$.

Since

$$MU_F^1 = \frac{1}{F_1}$$

$$MU_C^1 = \frac{2}{C_1}$$

$$MU_F^2 = \frac{2}{F_2}$$

$$MU_C^2 = \frac{1}{C_2}$$

then

$$MRS_{CF}^1 = \frac{2F_1}{C_1} \tag{1-7}$$

$$MRS_{CF}^2 = \frac{F_2}{2C_2} \tag{1-8}$$

Efficiency in exchange requires

$$\frac{2F_1}{C_1} = \frac{F_2}{2C_2}$$

and adding in overall efficiency means that

$$\frac{2F_1}{C_1} = \frac{F_2}{2C_2} = 2 \tag{1-9}$$

or

$$F_1 = C_1 \tag{1-10}$$

$$F_2 = 4C_2 \tag{1-11}$$

Since $C_1 + C_2 = C$ and $F_1 + F_2 = F$, equations (1-10) and (1-11) imply that (substituting $C - C_1$ for C_2 and $F - F_1$ for F_2 in (1-11))

$$C_1 = \frac{4C - F}{3} \tag{1-12}$$

Any production plan (F, C) for food and clothing which satisfies (1-6) will be efficient in production. Any division of the clothing from that plan which satisfies (1-12) (and (1-9)) will be efficient in exchange, and overall efficient.

So here is a recipe for all the efficient allocations :

i pick a total clothing supply, C

ii allocate C units of labour, and C units of machinery to the clothing industry

iii allocate the remaining $240 - C$ units of labour, and $120 - C$ units of capital, to the food industry ; that results in $F = 360 - 2C$ units of food being produced

iv allocate $C_1 = \frac{4C-F}{3}$ ($= 2C - 120$) units of clothing to person 1, and the remaining $C - \frac{4C-F}{3} = 120 - C$ units of clothing to person 2

v allocate $F_1 = C_1 = 2C - 120$ units of food to person 1, and $4C_2 = 480 - 4C$ units of food to person 2

vi notice that C_1 and F_1 defined in conditions *iv* and *v* are non-negative only if $C \geq 60$; so the total clothing supply defined in stage *i* leads to a sensible allocation only if $60 \leq C \leq 120$

What about the “other” part of the PPF, in which all the capital is allocated to clothing production, and in which the MRT increases from 2 to $2\sqrt{2}$? Could there be an overall efficient allocation in which production was on that part of the PPF?

Let p be the slope of the PPF. In this part of the PPF (in which $k_F = 0$), $p > 2$. Since $MRS^1 = MRS^2 = p$, if an efficient allocation were on this part of the PPF, we would have

$$MRS^1 = 2 \frac{F_1}{C_1} = p \quad (1 - 13)$$

$$MRS^2 = \frac{1}{2} \frac{F_2}{C_2} = p \quad (1 - 14)$$

Equation (1 - 13) implies $F_1 \geq C_1$ (if $p \geq 2$), and equation (1 - 14) implies that $F_2 \geq 4C_2 > C_2$. So together, they imply that

$$F = F_1 + F_2 > C_1 + C_2 = C$$

But we cannot have $F > C$ on this part of the PPF; if $k_F = 0$, then $F < 120 < C$. So overall efficiency requires that the aggregate production combination be on the straight part of the PPF, so that steps (i)–(vi) above describe all the Pareto optimal allocations in this economy.

Q2. There are 2 people in the economy. Each person’s preferences can be represented by the same utility function :

$$u = c - \frac{1}{2}n^2$$

where c is the person’s consumption, and n is the quantity of labour that she provides.

The consumption good is produced by labour alone. Person 1 can produce 1 unit of the consumption good per unit of her labour provided, and person 2 can produce 2 units the consumption good per unit of labour provided. (So the only difference between people is that person 2’s labour is twice as productive.)

What are all the Pareto optimal allocations in this economy?

A2. If person 1 provides n_1 units of labour, and person 2 provides n_2 , then the total quantity produced of the consumption good is

$$C = n_1 + 2n_2 \quad (2 - 1)$$

Suppose that a benevolent dictator chose labour supplies n_1 and n_2 for the 2 people, and consumption levels c_1, c_2 for each of them, to maximize some welfare function $W(u_1, u_2)$, where $W(u_1, u_2)$ is an increasing function of the two people’s utilities. Equation (2 - 1) implies that $c_2 = n_1 + 2n_2 - c_1$

if the allocation is feasible. Substituting this constraint, the dictator chooses c_1 , n_1 and n_2 to maximize

$$W\left[c_1 - \frac{1}{2}n_1^2, n_1 + 2n_2 - c_1 - \frac{1}{2}n_2^2\right] \quad (2-2)$$

The first-order conditions for this maximization are

$$W_1 - W_2 = 0 \quad (2-3)$$

$$-W_1 n_1 + W_2 = 0 \quad (2-4)$$

$$W_2[2 - n_2] = 0 \quad (2-5)$$

(corresponding respectively to setting the partial derivatives of (2-2) with respect to c_1 , n_1 and n_2 equal to zero), where

$$W_1 \equiv \frac{\partial W}{\partial u_1}$$

$$W_2 \equiv \frac{\partial W}{\partial u_2}$$

An immediate implication of equation (2-5) is that $n_2 = 2$ in any Pareto optimal allocation. Substituting $W_1 = W_2$ from equation (2-3) into equation (2-4) implies $n_1 = 1$.

So in any Pareto optimal allocation, person 1 should supply 1 unit of labour, and person 2 should supply 2.

The resulting output is 5 units of C . These units can be divided between the two people in any possible fashion : ranging from (0, 5) for an allocation which is most beneficial to person 2, to (5, 0) for the allocation which is most beneficial to person 1.

The same result could be obtained using the fundamental theorems of welfare economics. If the price of the consumption good is normalized to 1, then in any competitive equilibrium, person 1's wage is his marginal productivity, 1, and person 2's wage is her marginal productivity, 2. Given these wages, person 1 will maximize utility (whatever his endowment) by supplying 1 unit of labour, and person 2 will maximize her utility by supplying 2 units.

As well, the same outcome could be derived by considering the results of a lump-sum transfer of T units of the consumption good from person 2 to person 1. Given such a transfer, person 1's utility-maximizing bundle is $(T + 1, 1)$ and person 2's is $(4 - T, 2)$, where the bundles are (c_1, n_1) and (c_2, n_2) respectively. As T varies from 4 to -1 , all the Pareto optimal allocations are obtained.

Q3. Suppose now that, in the economy described in question #2 above, there were not 2 people, but 2 million, 1 million of each type. Otherwise the technology and preferences are exactly as described in question #2.

If the government's only possible tax were a proportional wage income tax — at the same rate on each person's wage income — what tax rate would type-1 people prefer, if the government returns its tax revenue to the 2 million people, dividing it equally among them?

A3. Let τ be the proportional tax rate on each person's labour income. Let the price of the consumption good be 1. Then the equilibrium wage rates are 1 for person 1, and 2 for person 2, since that is how much output they each produce per unit of labour supplied.

So, taken into account the tax rates they face, person 1 faces a net wage of $1 - \tau$ per unit labour supplied, and person 2 faces a net wage rate of $2(1 - \tau)$ per unit of labour she supplies.

In a competitive economy, person 1's consumption is his wage income, plus his share of government tax revenue. So he chooses his labour supply n_1 to maximize

$$u_1 = (1 - \tau)n_1 + s - \frac{1}{2}n_1^2 \quad (3 - 1)$$

where s is his share of the government tax revenue. Choosing n_1 to maximize (3 - 1) means choosing n_1 so that

$$(1 - \tau) = n_1 \quad (3 - 2)$$

[Notice that the person's labour supply here is independent of his exogenous income s , since he has *quasi-linear* preferences.]

Similarly, person 2 chooses n_2 to maximize $u_2 = 2(1 - \tau)n_2 + s - \frac{1}{2}n_2^2$, resulting in an optimal labour supply of $n_2 = 2(1 - \tau)$.

How much revenue is the government collecting from the tax? Each type-1 person is earning $1 - \tau$, since he chooses to supply $1 - \tau$ units of labour, at a wage of 1, and each type-2 person is earning $4(1 - \tau)$, since she is paid 2 per unit labour supplied, and supplying $2(1 - \tau)$ units of labour. Since the government is collecting a proportion τ of all this revenue, its total tax collections are 1 million times

$$\tau[(1 - \tau) + 4(1 - \tau)] = 5\tau(1 - \tau) \quad (3 - 3)$$

Each person collects an equal share of this revenue, which means that

$$s = \frac{5}{2}\tau(1 - \tau) \quad (3 - 4)$$

So if the tax rate is τ , a type-1 person will choose to supply $1 - \tau$ units of labour, will collect earnings (net of tax) of $(1 - \tau)^2$ from that labour supply, and will also get a payment of $s = \frac{5}{2}\tau(1 - \tau)$ from the government. So his consumption level will be his labour earnings, plus s , or

$$c_1 = (1 - \tau)^2 + \frac{5}{2}\tau(1 - \tau) = (1 - \tau)\left(1 + \frac{3}{2}\tau\right) \quad (3 - 5)$$

The best tax rate τ is the one which gives him the highest level of utility $c_1 - \frac{1}{2}n_1^2$. Given that $n_1 = 1 - \tau$, his utility is

$$u_1 = c_1 - \frac{1}{2}n_1^2 = (1 - \tau)\left(1 + \frac{3}{2}\tau\right) - \frac{1}{2}(1 - \tau)^2 = \frac{1}{2}(1 - \tau)(1 + 4\tau) \quad (3 - 6)$$

Differentiating (3 - 6) with respect to τ , and setting equal to 0, yields an optimal tax rate (from the perspective of type-1 people) of $3/8$. Even though the wage tax redistributes from type-2

people to type-1 people, increasing the tax rate here above $3/8$ makes the type-1 people worse off, because it reduces labour supply so much.

Q4. If the initial prices of goods 1, 2 and 3 are all 1, and if a person's expenditure function can be written

$$E(p_1, p_2, p_3, u) = p_1 u + 3(p_1 p_2 p_3)^{1/3}$$

then what is the excess burden of a tax system which taxes good 2 at a proportional rate of τ_2 , good 3 at a proportional rate of τ_3 , and does not tax good 1 (so that the tax-inclusive prices of the three goods are 1, $1 + \tau_2$ and $1 + \tau_3$)?

A4. The microeconomic trick to use here is *Shephard's Lemma*, that the compensated demand for good i is the derivative of the person's expenditure function with respect to the price of good i . Given the above expenditure function, its derivatives with respect to the prices of goods 2 and 3 are

$$E_2(p_1, p_2, p_3, u) \equiv \frac{\partial E}{\partial p_2} = p_1^{1/3} p_2^{-2/3} p_3^{1/3} \quad (4-1)$$

$$E_3(p_1, p_2, p_3, u) \equiv \frac{\partial E}{\partial p_3} = p_1^{1/3} p_2^{1/3} p_3^{-2/3} \quad (4-2)$$

So if the prices of the 3 goods start out at $(1, 1, 1)$, and then the government levies a proportional tax at the rate of τ_2 on good 2, and τ_3 on good 3 – and compensates the person so that she stays at her original level u of utility — then the person's quantities demanded of goods 2 and 3 are $E_2(1, 1 + \tau_2, 1 + \tau_3, u)$ and $E_3(1, 1 + \tau_2, 1 + \tau_3)$ respectively. [Why didn't I bother with the compensated demand for good 1? Because the government will not be collecting any tax revenue on good 1.]

The government's tax collections are τ_2 times the quantity demanded of good 2, plus τ_3 times the quantity demanded of good 3 :

$$T = \tau_2[(1 + \tau_2)^{-2/3}(1 + \tau_3)^{1/3}] + \tau_3[(1 + \tau_2)^{1/3}(1 + \tau_3)^{-2/3}] = \left(\frac{\tau_2}{1 + \tau_2} + \frac{\tau_3}{1 + \tau_3}\right)(1 + \tau_2)^{1/3}(1 + \tau_3)^{1/3} \quad (4-3)$$

(where I have used the facts that $p_1 = 1$, $p_2 = 1 + \tau_2$ and $p_3 = 1 + \tau_3$.)

At the original (no-tax) prices of $(1, 1, 1)$, the value of the person's expenditure function is $u + 3$. After the taxes, the value of her expenditure function is

$$E(1, 1 + \tau_2, 1 + \tau_3, u) = u + 3(1 + \tau_2)^{1/3}(1 + \tau_3)^{1/3} \quad (4-4)$$

The excess burden of the tax system is $E(1, 1 + \tau_2, 1 + \tau_3, u) - E(1, 1, 1, u)$, minus the tax revenue collected. So

$$EB = 3(1 + \tau_2)^{1/3}(1 + \tau_3)^{1/3} - 3 - \left(\frac{\tau_2}{1 + \tau_2} + \frac{\tau_3}{1 + \tau_3}\right)(1 + \tau_2)^{1/3}(1 + \tau_3)^{1/3}$$

or

$$EB = (1 + \tau_2)^{1/3}(1 + \tau_3)^{1/3} \left[3 - \left(\frac{\tau_2}{1 + \tau_2} + \frac{\tau_3}{1 + \tau_3}\right)\right] - 3 \quad (4-5)$$

Q5. For the person described in question #4 above, what tax rate τ_3 on good 3 will minimize the excess burden of the tax system, if good 1 is not taxed, and good 2 is taxed at a rate of 50 percent?

A5. If good 2 is taxed at fifty percent, so that $\tau_2 = 0.5$, equation (4 – 5) becomes

$$EB = (1.5)^{1/3}(1 + \tau_3)^{1/3}\left[3 - \frac{1}{3} - \frac{\tau_3}{1 + \tau_3}\right] - 3 \quad (5 - 1)$$

Differentiating,

$$\frac{\partial EB}{\partial \tau_3} = (1.5)^{1/3}\left[\frac{1}{3}\frac{8}{3}(1 + \tau_3)^{-2/3} - (1 + \tau_3)^{-2/3} + \frac{2}{3}\tau_3(1 + \tau_3)^{-5/3}\right] \quad (5 - 2)$$

or

$$\frac{\partial EB}{\partial \tau_3} = (1.5)^{1/3}\frac{1}{9}(1 + \tau_3)^{-5/3}[6\tau_3 - (1 + \tau_3)] \quad (5 - 3)$$

Equation (5 – 3) shows that, if initially $\tau_1 = 0$ and $\tau_2 = 0.5$, increasing τ_3 slightly above zero will actually reduce the overall excess burden, since it is a net substitute for good 2, which is being taxed at a positive rate.

But expression (5 – 3) is negative only if $6\tau - (1 + \tau) < 0$, or $\tau < 1/5$.

Here, given that $\tau_1 = 0$ and $\tau_2 = 0.5$, the overall excess burden is minimized by setting $\tau_3 = 0.2$, because that tax of twenty percent makes expression (5 – 3) equal to zero.