GS/ECON 5300   PUBLIC ECONOMICS I

Answers to Midterm Exam   February 2006

$Q1$. What is the effect on a person's saving of a tax on the return to saving to the following person? The person lives for two periods, earns all of her income in the first period and has preferences which can be represented by a utility function

$$U(C_1, C_2) = \sqrt{C_1} + \sqrt{C_2}$$

defined over consumption in the first and second period of her life. Here C_1 is her first–period consumption, and C_2 is her second–period consumption.

$A1$. First, find a person's demand function for good 1, if she had an exogenous income of Y, and wanted to maximize $U(C_1, C_2)$ with respect to C_1 and C_2, subject to the budget constraint $p_1C_1 + p_2C_2 = Y$. Tangency of her indifference curve requires that her MRS equal the price ratio. Here, since

$$MU_1 = \frac{1}{2\sqrt{C_1}}$$

$$MU_2 = \frac{1}{2\sqrt{C_2}}$$

this tangency condition is

$$\sqrt{\frac{C_1}{C_2}} = \frac{p_2}{p_1} \tag{1-1}$$

Equation $(1-1)$ implies that

$$C_2 = (\frac{p_1}{p_2})^2 C_1$$

so that the person's budget constraint can be written

$$p_1C_1 + p_2(\frac{p_1}{p_2})^2 C_1 = Y$$

or

$$C_1 = \frac{p_2}{p_1}\frac{Y}{p_1 + p_2} \tag{1-2}$$

Now let period–1 consumption be the numéraire. So the present value of her lifetime income, expressed in terms of period–1 consumption is W, the income she earns in period 1. Setting $p_1 = 1$, since good 1 is the numéraire, $(1-2)$ becomes

$$C_1 = \frac{p_2 W}{1 + p_2} \tag{1-3}$$

implying that

$$\frac{\partial C_1}{\partial p_2} = \frac{1}{(1 + p_2)^2} W > 0 \tag{1-4}$$

The price of second–period consumption, in terms of first–period consumption, is

$$p_2 = \frac{1}{1 + r(1 - \tau)}$$

where r is the gross–of–tax rate of return on saving, and τ is the tax rate on the return to saving.

That means that p_2 is an increasing function of τ. Since saving equals $W - C_1$, therefore saving decreases with the tax rate τ (since C_1 increases with p_2, which increases with τ).

Here the preferences are CES, with a constant elasticity of substitution of $1/(1 - \rho) = 2$. Since the elasticity of substitution is high, here it dominates the income effect, so that saving must rise with its net return.

*Q*2. What is the excess burden of a tax on clothing consumption, if there are no other taxes, in a one–consumer economy in which the one consumer's preferences can be represented by the utility function

$$U(x_0, x_, x_2, \ldots, x_n) = x_0 + a_1 \ln x_1 + a_2 \ln x_2 + \cdots + a_n \ln x_n$$

where x_i is her consumption of good i, where good #1 is clothing, and where the producer price of each good is 1?

*A*2. In this case, the first–order conditions for utility maximization by the consumer are

$$1 = \lambda P_0 \qquad (2-1)$$

$$\frac{a_i}{x_i} = \lambda P_i \quad i = 1, 2, \ldots, n \qquad (2-2)$$

where λ is the Lagrange multiplier on the budget constraint $P_0 x_0 + P_1 x_1 + \cdots + P_n x_n = Y$, and where P_i is the price the consumer faces for good i.

Equations $(2-1)$ and $(2-2)$ imply that the demand function for good i is

$$x_i = a_i \frac{P_0}{P_i} \qquad (2-3)$$

Demand for good i ($i > 0$) does not depend on the consumer's income, since her preferences are quasi–linear.

The Hicksian demand for good 0 can be found from the fact that

$$u = x_0 + a_1 \ln \frac{a_1 P_0}{P_1} + a_2 \ln \frac{a_2 P_0}{P_2} + \cdots + a_n \ln \frac{a_n P_0}{P_n} \qquad (2-4)$$

implying that

$$x_0^H(P_0, P_1, \ldots, P_n, u) = u - a_1 \ln \frac{a_1 P_0}{P_1} - a_2 \ln \frac{a_2 P_0}{P_2} - \cdots - a_n \ln \frac{a_n P_0}{P_n} \qquad (2-5)$$

which means that the expenditure function is

$$e(P_0, P_1, \ldots, P_n, u) = P_0 u - P_0[a_1 \ln \frac{a_1 P_0}{P_1} + a_2 \ln \frac{a_2 P_0}{P_2} + \cdots + a_n \ln \frac{a_n P_0}{P_n}] + P_0[a_1 + a_2 + \cdots + a_n] \qquad (2-6)$$

Now the only good taxed is good 1, so that

$$P_1 = p_1 + t_1$$

where t_1 is the unit tax on good 1, and p_1 is the net–of–tax price.

The excess burden of the tax is defined as

$$EB = e(P_0, p_1 + t_1, P_2, \ldots, P_n, u) - e(P_0, p_1, P_2, \ldots, P_n, u) - t_1 a_1 \frac{P_0}{P_1}$$

which (from equation $(2-6)$) equals

$$EB = P_0 a_1[\ln(p_1 + t_1) - \ln p_1] - t_1 a_1 \frac{P_0}{p_1 + t_1} \qquad (2-7)$$

As it must be, the excess burden is positive if $t_1 \neq 0$. $EB = 0$ when $t_1 = 0$, and

$$\frac{\partial EB}{\partial t_1} = t_1 \frac{a_1 P_0}{(p_1 + t_1)^2} \qquad (2-8)$$

which is positive if and only if $t_1 > 0$.

*Q*3. Why should the marginal tax rate be 0 on the highest observed level of income?

*A*4. There are (at least) 2 possible explanations.

One, the formal proof. Let

$$U^i(C, Y) = \tilde{U}(C, 1 - \frac{Y}{w_i})$$

be the utility of a person of type i, if she has a gross wage of w_i, and has to earn Y, and gets to consume C. (U is a function of consumption and of leisure.)

Then, with 2 types of people, the optimal income tax problem is to maximize some welfare function $W[\tilde{U}^1(C_1, Y_1), \tilde{U}^2(C_2, Y_2)]$ subject to the feasibility constraint $Y_1 + Y_2 - C_1 - C_2 \geq \bar{R}$, where $\bar{R}$ is the exogenous revenue requirement for the tax, and the selction constraint, that the high–ability type–2 person does not want to pick the low–ability person's income and consumption : $\tilde{U}^2(C_2, Y_2) \geq \tilde{U}^2(C_1, Y_1)$.

The resulting Lagrangean is

$$W[\tilde{U}^1(C_1, Y_1), \tilde{U}^2(C_2, Y_2)] + \lambda[Y_1 + Y_2 - C_1 - C_2 - \bar{R}] + \mu[\tilde{U}^2(C_2, Y_2) - \tilde{U}^2(C_1, Y_1)]$$

The first–order conditions with respect to C_2 and Y_2 are

$$(W_2 + \mu)\tilde{U}^2_C - \lambda = 0 \qquad (3-1)$$

$$(W_2 + \mu)\tilde{U}^2_Y + \lambda = 0 \qquad (3-2)$$

where W_i is the derivative of the welfare function with respect to the utility of type–i people. Equations $(3-1)$ and $(3-2)$ imply that

$$\frac{\tilde{U}^2_Y}{\tilde{U}^2_C} = -1 \qquad (3-3)$$

Now if a person of type 2 chooses how much to earn and consume, she maximizes $\tilde{U}^2(Y - T(Y), Y)$, where $T(Y)$ is the total income tax payable on an income of Y, with first–order condition

$$\frac{\tilde{U}^2_Y}{\tilde{U}^2_C} = -(1 - T'(Y)) \qquad (3-4)$$

Comparing equations $(3-3)$ and $(3-4)$, the optimal income tax requires $T'(Y_2) = 0$: a zero marginal rate at the level of income earned by the richer of the people.

A second proof, is more of a logical proof. Suppose, contrary to the assertion in the question, that the marginal rate was positive on the income earned by the highest income earner in the country. Suppose that this person's income was $\tilde{Y}$. Consider now the effect of lowering the marginal rate on income above $\tilde{Y}$, leaving the tax schedule unchanged on income levels in $[0, \tilde{Y})$. This change will not affect the labour choices of people earning less than the maximum income : their marginal rates are unchanged. It can't induce the highest income earner to work less : her tax schedule is unchanged to the left of $\tilde{Y}$. But it might induce her to work more. If so, more income tax revenue will be collected, since the marginal rate is positive to the right of $\tilde{Y}$, and richest person's taxable income has increased above $\tilde{Y}$. The richest person must also be at least as well off : her budget set has been expanded by the marginal rate cut.

Therefore, if the marginal rate were positive on income just above the highest observed income, then a decrease in this rate will raise at least as much revenue, and make everyone at least as well off. Continuing the tax cutting, this process implies that it cannot be efficient to have a positive marginal tax rate at and above the highest observed income, since cutting the marginal rate would then imply a Pareto improvement.