*Q*1. If individuals had the same preferences, and differed only in "ability" (the amount of output that a person can produce per hour), illustrate what outcomes can be achieved (*i*) by lump–sum taxation based on people's abilities and (*ii*) by an income tax levied on people's (endogenous) labour income (under which each person is given the same income tax schedule).

*A*4. Figure 11–5 (pg. 359) in *Atkinson and Stiglitz* is an example of the illustration of the trade–offs in the two situations. [Figure 1 below is similar.]

When lump–sum taxation is possible, the "utility possibility frontier" slopes down. Increases in the lump–sum tax on person 2 (and therefore in the lump–sum grant received by person 1) will increase person 1's utility (by his marginal utility of income) and decrease person 2's (by her marginal utility of income).This first–best utility frontier will cross the 45–degree line, because it is possible to make the low–ability person better off than the high–ability person, by making a large enough transfer fromthe high–ability person tot he low–ability person.

Compared with the first–best utility possibility frontier, there are three main features of the "second–best utility possibility frontier, attainable when an income tax is used (if both people must be given the identical income tax schedule),as compared to the first–best upf.

First of all, the second–best upf must be inside (or on) the first–best upf. The second–fundamental theorem of welfare economics shows that no outcome under income taxation can be better for both people than any outcome using lump–sum taxation.

Secondly, zero taxes are certainly feasible in a second–best world.The first fundamental theorem of welfare economics shows that the outcome with no taxes must also be Pareto optimal. So there is some utility combination (marked N in the figures) which is on both the first–best and second–best upfs, corresponding to no taxes at all. The two upfs are tangent at this point.

Thirdly, the second–best upf cannot go below the 45–degree line (if the utility of the high–ability person is graphed on the vertical axis). Because the income tax schedule is "anonymous", person 2 always has the option of choosing to earn the exact same income as person 1 chooses. That would give her the same after–tax income, and thus the same consumption. But since person 2 has higher ability, she would not need to work as long to earn the same amount of money, so that she would have a higher amount of leisure if she chose to earn the same income,. Thus, whatever person 1 chooses, person 2 always has a feasible option which makes her better off. Therefore, person 2 must always be better off than person 1, sothat the second–best upf must stay above the 45–degree line in this case.

Because the second–best upf stays above the 45–degree line, it must eventually "turn back in" as we move down and to the right. Beyond this point, further increases in marginal tax rates will actually make both people worse off. The well–being of the low–ability person is maximized at the rightmost point on the second–best upf, where the curve is vertical before turning back in towards the origin. A "Rawlsian", or "max–min" welfare measure would choose the tax policy leading to

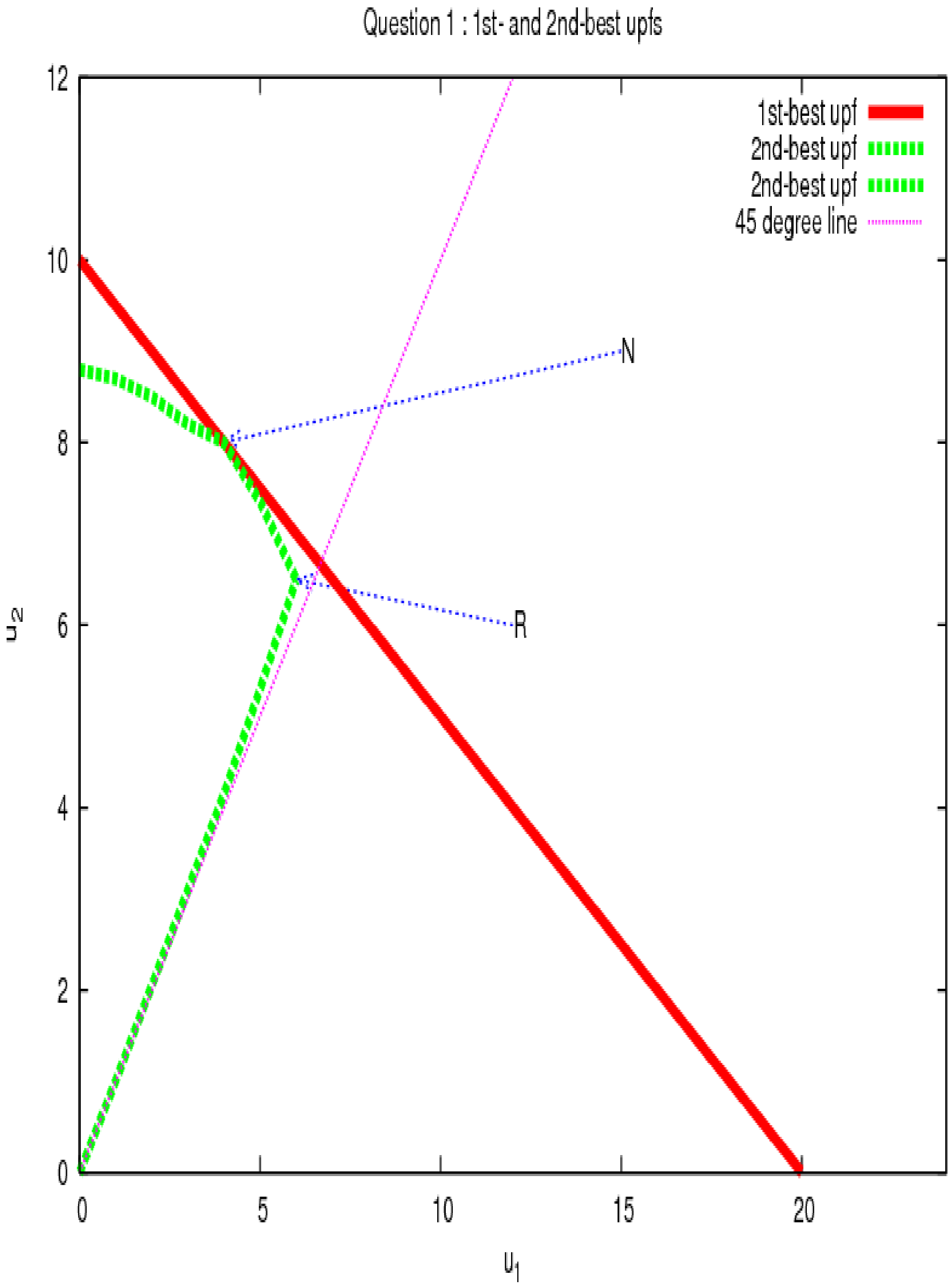

Figure 1 : the utility possibility frontiers

this outcome (marked “R” in figure 1): any further move towards greater equality will harm both people.

*Q*2. Derive a formula for the excess burden of a commodity tax system which taxes good 1 at the (proportional) rate τ_1, and good 2 at the (proportional) rate τ_2, and leaves good 3 untaxed, if the taxpayer’s expenditure function were

$$E(q_1, q_2, q_3, u) = q_1^a q_2^b q_3^{1-a-b} u$$

where a and b are positive parameters with $a + b < 1$, if the producer prices of the goods were $p_1 = p_2 = p_3 = 1$.

*A*2. If the producer prices (not including taxes) of goods are denoted (p_1, p_2, p_3), and the consumer prices (including taxes) are denoted (q_1, q_2, q_3), then the excess burden of a tax system which levies proportional taxes at the rate τ_i on good i is

$$E(q_1, q_2, q_3, u) - E(p_1, p_2, p_3, u) - \sum_{i=1}^{3} \tau_i p_i x_i^c(q_1, q_2, q_3, u)$$

where $E(\cdot, \cdot, \cdot, \cdot)$ is the expenditure function, u is the person’s reference level of utility, and x_i^c is her compensated or “Hicksian” demand function for good i.

Shephard’s Lemma implies that the compensated demand for good i is the partial derivative of the expenditure function with respect to the price of that good. Given that $E(q_1, q_2, q_3) = q_1^a q_2^b q_3^{(}1 - a - b)u$,

$$x_1^c(q_1, q_2, q_3, u) = \frac{a}{q_1} E(q_1, q_2, q_3)$$

$$x_2^c(q_1, q_2, q_3, u) = \frac{b}{q_2} E(q_1, q_2, q_3)$$

Since the producer prices p_i are all equal to 1, then

$$q_1 = 1 + \tau_1$$

$$q_2 = 1 + \tau_2$$

$$q_3 = 1$$

Therefore the tax revenue collected, $\tau_1 x_1^c + \tau_2 x_2^c$, equals

$$TR = [a\frac{\tau_1}{1 + \tau_1} + b\frac{\tau_2}{1 + \tau_2}]E(q_1, q_2, q_3)$$

and the excess burden equals

$$E(q_1, q_2, q_3) - E(p_1, p_2, p_3, u) - [a\frac{\tau_1}{1 + \tau_1} + b\frac{\tau_2}{1 + \tau_2}]E(q_1, q_2, q_3)$$

or

$$(1+\tau_1)^a(1+\tau_2)^b(1-[a\frac{\tau_1}{1+\tau_1}+b\frac{\tau_2}{1+\tau_2}])u-u$$

If u is the initial level of the person's utility, then $E(p_1,p_2,p_3,u)=y$, the person's income, so that the excess burden also would equal

$$(1+\tau_1)^a(1+\tau_2)^b(1-[a\frac{\tau_1}{1+\tau_1}+b\frac{\tau_2}{1+\tau_2}])y-y$$

*A*3. What should the marginal personal income tax rate be at the highest observed income in a country? Explain briefly.

*A*3. The tax rate, right at and above the highest observed income level, should be zero.

Two possible explanations.

One, suppose that, to the contrary, the tax rate were positive above the highest observed income. Then lowering that rate — above, but not below the highest observed income — would make the highest–income person better off, since it expands her budget set.

It does not affect directly the well–being of anyone choosing to earn lower levels of income, since their tax schedules have not really been changed.

But the change will generate more tax revenue, the revenue on the additional income the highest–income person chooses to earn. So the change is a Pareto improvement since it generates more revenue, and makes no–one worse off. This implies the original tax schedule, in which the marginal rate was positive at the top, could not have been optimal.

A mathematical proof is to write a person's well being as $u(c,y)$ where c is the person's consumpion, and y the income she earns. Earning more income lowers u. The person, facing a tax schedule implying $c=y-T(y)$ (where $T(y)$ is the total tax payable on an income of y, maximizes $u(y-T(y),y)$, and so has a first–order condition for optimality

$$(1-T'(y))u_c+u_y=0 \qquad (3-1)$$

Now consider the government's optimal income tax problem in a two–person world, choosing consumption levels and incomes for the two people, subject to the budget constraint that total income exceed total consumption plus the government's revenue requirement, but also the additional constraint that the high–income person cannot prefer to "mimic" the low–ability person by choosing his income and consumption. Formally, this problem is to maximize some welfare function

$$W(u^1(c_1,y_1),u^2(c_2,y_2))$$

subject to the two constraints

$$y_1+y_2-c_1-c_2\geq R$$

$$u2(c_2, y_2) \geq u^2(c_1, y_1)$$

where person 2 is the high–income person, and R is the government's revenue requirement.

The first–order conditions for this problem are

$$W_1 u_c^1(c_1, y_1) - \lambda + \mu u_c^2(c_1, y_1) = 0 \qquad (3-2)$$

$$W_1 u_y^1(c_1, y_1) + \lambda + \mu u_y^2(c_1, y_1) = 0 \qquad (3-3)$$

$$W_2 u_c^2(c_2, y_2) - \lambda - \mu u_c^2(c_2, y_2) \qquad (3-4)$$

$$W_2 u_y^2(c_2, y_2) + \lambda - \mu u^2(c_2, y_2) \qquad (3-5)$$

where λ is the Lagrange multiplier on the aggregate resource constraint and μ is the Lagrange multiplier on the "no mimicry" constraint.

Adding up equations $(3-4)$ and $(3-5)$,

$$[W_2 - \mu][u_c^2(c_2, y_2) + u_y^2(c_2, u_2)] = 0 \qquad (3-6)$$

But person 2 actually chooses (c_2, y_2) from a given tax schedule, meaning that equation $(3-1)$ holds,and can be written

$$u_c^2(c_2, y_2) + u_y^2(c_2, u_2) = T'(y_2) u_c^2(c_2, y_2) \qquad (3-7)$$

Equations $(3-6)$ and $(3-7)$ therefore show that $T'(y_2)$, the marginal income tax rate at the highest observed income, must equal zero.