

Q1. What does the contract curve look like for a 2-person, 2-good exchange economy, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = 100 - \frac{1}{x_1^1} - \frac{1}{x_2^1}$$

$$u^2(x_1^2, x_2^2) = x_2^2 + 100 \ln x_1^2$$

where x_j^i is person i 's consumption of good j ?

A1. The two people's marginal rates of substitution are

$$MRS^1 = \frac{u_1^1}{u_2^1} = \frac{(x_2^1)^2}{(x_1^1)^2}$$

$$MRS^2 = \frac{u_1^2}{u_2^2} = \frac{100}{x_1^2}$$

So an interior allocation will be on the contract curve if $MRS^1 = MRS^2$, or

$$\frac{(x_2^1)^2}{(x_1^1)^2} = \frac{100}{x_1^2} \quad (1-1)$$

Given that

$$x_1^2 = e_1 - x_1^1$$

where e_j is the total endowment of good j , equation (1-1) can be written

$$x_2^1 = 10 \frac{x_1^1}{\sqrt{e_1 - x_1^1}} \quad (1-2)$$

which defines an upward-sloping curve in x_1^1 - x_2^1 space.

The contract curve goes through the bottom left corner of the Edgeworth box : when $x_1^1 = 0$, equation (1-2) indicates that $x_2^1 = 0$. However, the curve defined by equation (1-2) hits the top edge of the Edgeworth box : as $x_1^1 \rightarrow e_1$, $x_2^1 \rightarrow \infty$ according to (1-2).

The contract curve hits the top of the Edgeworth box when $x_2^1 = e_2$, or

$$e_2 = 10 \frac{x_1^1}{\sqrt{e_1 - x_1^1}} \quad (1-3)$$

Equation (1-3) can be solved for the value of x_1^1 at which $x_2^1 = e_2$ along the contract curve. Using the quadratic formula

$$x_1^1 = \frac{e_2 \sqrt{(e_2)^2 + 400e_1} - (e_2)^2}{200} \equiv \bar{x}_1 \quad (1-4)$$

So the contract curve consists of all (x_1^1, x_2^1) pairs satisfying equation (1 – 2), with $0 \leq x_1^1 \leq \bar{x}_1$, and then the set of all (x_1^1, x_2^1) with $\bar{x}_1 < x_1^1 \leq e_1$, and $x_2^1 = e_2$.

Q2. What are all the allocations in the core of a 3–person, 2–good economy, in which each person has the same endowment vector, $e^i = (1, 1)$, and in which the preferences of the 3 people can be represented by the utility functions listed below?

$$u^1(x_1^1, x_2^1) = x_1^1$$

$$u^2(x_1^2, x_2^2) = x_2^2$$

$$u^3(x_1^3, x_2^3) = x_1^3 + x_2^3$$

A2. If each person just consumed her endowment, $(1, 1)$, then people 1 and 2 would each have utility of 1, and person 3 would have utility of 2. So if an allocation is in the core, it must be “individually rational” : it must give people at least as high utility as they could get from their own endowments. So an allocation will be in the core only if

$$x_1^1 \geq 1 \tag{2 – 1}$$

$$x_2^2 \geq 1 \tag{2 – 2}$$

$$x_1^3 + x_2^3 \geq 2 \tag{2 – 3}$$

Further, the allocation must be Pareto optimal if it is in the core. (Otherwise it could be blocked by a coalition of all three people.) Given that person 3 finds the 2 goods to be perfect substitutes, that person 1 likes only good 1, and that person 2 likes only good 2, an allocation $(x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3)$ will be Pareto optimal (and feasible) only if

$$x_1^1 + x_1^2 + x_1^3 = 3 \tag{2 – 4}$$

$$x_2^1 + x_2^2 + x_2^3 = 3 \tag{2 – 5}$$

$$x_2^1 = 0 \tag{2 – 6}$$

$$x_1^2 = 0 \tag{2 – 7}$$

But people 1 and 2 could form a coalition of their own, and each get a utility of 2, by combining their endowments to get the allocation in which $\mathbf{x}^1 = (2, 0)$ and $\mathbf{x}^2 = (0, 2)$. So an allocation will be in the core only if it gives person 1 and person 2 each a utility of 2 or more. Given their utility functions, it must be the case then that an allocation is in the core only if

$$x_1^1 \geq 2 \tag{2 – 8}$$

$$x_2^2 \geq 2 \quad (2 - 9)$$

But combining (2 – 8) and (2 – 9) with person 3’s individual rationality requirement (2 – 3), and with the feasibility constraints (2 – 4) and (2 – 5), there is only one allocation left in the core : $\mathbf{x}^1 = (2, 0)$, $\mathbf{x}^2 = (0, 2)$ and $\mathbf{x}^3 = (1, 1)$.

Q3. How would the equilibrium prices of the goods vary with the people’s endowments in a 2–person, 2–good exchange economy, if each person’s preferences could be represented by the utility function

$$u^i(\mathbf{x}^i) = -\frac{1}{x_1^i} - \frac{1}{x_2^i}$$

where x_j^i was person i ’s consumption of good j ?

A3. Each person’s *MRS* is

$$MRS^i = \frac{u_1^i}{u_2^i} = \frac{(x_2^i)^2}{(x_1^i)^2} \quad (3 - 1)$$

Setting the *MRS* equal to the price ratio then implies that

$$\frac{(x_2^i)^2}{(x_1^i)^2} = \frac{p_1}{p_2} \quad (3 - 2)$$

or

$$x_2^i = \sqrt{p_1/p_2} x_1^i \quad (3 - 3)$$

Substitution of (3 – 3) into person i ’s budget constraint implies that

$$p_1 x_1^i + \sqrt{p_1 p_2} x_1^i = m^i \quad (3 - 4)$$

if m^i is person i ’s income, so that person i ’s demand function for good 1 can be written

$$x_1^i = \frac{m^i}{p_1 + \sqrt{p_1 p_2}} \quad (3 - 5)$$

(Since preferences here are CES, with $\rho = -1$, you can also get this demand function from the example in chapter 1 of the text.) But a person’s income is the value of her endowment :

$$m^i = p_1 e_1^i + p_2 e_2^i \quad (3 - 6)$$

so that the demand function (3 – 5) becomes

$$x_1^i = \frac{p_1 e_1^i + p_2 e_2^i}{p_1 + \sqrt{p_1 p_2}} \quad (3 - 7)$$

The market for good 1 will clear only if total demand equals the total endowments, or

$$x_1^1 + x_1^2 = \frac{p_1 e_1 + p_2 e_2}{p_1 + \sqrt{p_1 p_2}} = e_1 \quad (3 - 8)$$

(We do not need to consider market clearing in the market for the other good, since Walras's Law guarantees that if aggregate demand for good 1 equals the aggregate endowment of good 1, then the aggregate demand for good 2 will also equal the aggregate endowment of good 2.) But equation (3 – 8) is equivalent to

$$\frac{p_1}{p_2} = \frac{(e_2)^2}{(e_1)^2} \quad (3 - 9)$$

Equation (3 – 9) defines how the equilibrium price ratio varies with the people's endowments of the goods. Because people here have identical, homothetic preferences, the equilibrium prices depend only on the **aggregate** endowments, not on how the endowments are distributed between the 2 people.

Q4. Find all the pure-strategy Nash equilibria in the following strategic-form two-person game.

	<i>LL</i>	<i>L</i>	<i>R</i>	<i>RR</i>
<i>tt</i>	(4, 6)	(9, 3)	(2, 5)	(10, 1)
<i>t</i>	(1, 2)	(3, 4)	(4, 3)	(10, 2)
<i>b</i>	(2, 7)	(7, 2)	(5, 7)	(0, 0)
<i>bb</i>	(3, 5)	(8, 6)	(8, 8)	(12, 3)

A4. This game cannot be solved completely by iterated elimination of dominated strategies. But it can be partially solved that way.

Note first that column *RR* is a strictly dominated strategy for player 2. (It is dominated strictly by column *L*.) Once column *RR* is crossed out, row *t* of player 1 is dominated strictly by row *b* : so we can cross out row *t*.

Once row *t* is crossed out, column *L* is dominated strictly by column *R* for player 2, so that column *L* can be crossed out. With columns *L* and *RR* crossed out, row *b* is dominated by row *bb* for player 1. Crossing that row out reduces the game to the following 2-by-2 game

	<i>LL</i>	<i>R</i>
<i>tt</i>	(4, 6)	(2, 5)
<i>bb</i>	(3, 5)	(8, 8)

This new game has 2 pure-strategy Nash equilibria : (*tt*, *LL*) and (*bb*, *R*). These are the only pure-strategy equilibria to the whole game (since a strategy which has been crossed out during iterated elimination of strictly dominated strategies cannot be played in any pure-strategy (or mixed-strategy) Nash equilibrium).

There also is a mixed-strategy equilibrium to this game, in which player 1 plays *tt* with probability 3/4, *bb* with probability 1/4 and the other two strategies with probability 0, and in which player 2 plays *LL* with probability 6/7, *R* with probability 1/7, and the other 2 strategies

with probability 0. That's the only mixed-strategy equilibrium. But the question did not ask about mixed-strategy equilibria.

Q5. Find all the Nash equilibria (in pure or mixed strategies) to the following two-person game in strategic form.

	<i>L</i>	<i>R</i>
<i>t</i>	(12, 6)	(6, 4)
<i>m</i>	(0, 8)	(7, 12)
<i>b</i>	(2, 2)	(8, 4)

A5. Note first that *m* is a strictly dominated (by *b*) strategy for player 1. That means that *m* will not be played with positive probability by player 1. We can restrict attention to equilibria in which the other 2 strategies are played with positive probability.

There are two pure strategies to this game, (*t*, *L*) and (*b*, *R*).

To find any mixed strategies, consider what would induce player 1 to mix between strategies *t* and *b* (since we know she will not want to play *m* with any positive probability). If player 2 plays *L* with probability β , and *R* with probability $1 - \beta$, then player 1 will get the expected payoffs of $12\beta + 6(1 - \beta)$ from strategy *t* and $2\beta + 8(1 - \beta)$ from strategy *b*. She will be willing to randomize only if

$$12\beta + 6(1 - \beta) = 2\beta + 8(1 - \beta)$$

or

$$\beta = \frac{1}{6}$$

Player 2 will be willing to randomize between *L* and *R* if he gets the same expected payoffs from these 2 pure strategies. Since his expected payoffs from these two pure strategies are $6\alpha + 2(1 - \alpha)$ and $4\alpha + 4(1 - \alpha)$ respectively, if player 1 plays her pure strategy *t* with probability α , then player 2 will be willing to randomize only if

$$6\alpha + 2(1 - \alpha) = 4\alpha + 4(1 - \alpha)$$

or

$$\alpha = \frac{1}{2}$$

So there is a mixed strategy equilibrium in which player 1's mixing probabilities over her pure strategies are $(1/2, 0, 1/2)$, and in which player 2's mixing probabilities are $(1/6, 5/6)$.