Indirect Utility and Expenditure Functions: Some Examples

\( i \) Fixed Coefficients

If the utility function is

\[ u(x) = \min(ax_1, bx_2) \]

then the Marshallian demands are

\[ x_1 = \frac{b}{p_1b + p_2a} M \]

and

\[ x_2 = \frac{a}{p_1b + p_2a} M \]

so that

\[ ax_1 = bx_2 = \frac{ab}{p_1b + p_2a} M \]

which means that the indirect utility function is

\[ v(p_1, p_2, M) = \frac{ab}{p_1b + p_2a} M \]

A check on Roy’s Identity: from the above indirect utility function,

\[ \frac{\partial v}{\partial M} = \frac{ab}{p_1b + p_2a} \]

\[ \frac{\partial v}{\partial p_1} = -\frac{ab^2}{(p_1b + p_2a)^2} \]

\[ \frac{\partial v}{\partial p_2} = -\frac{a^2b}{(p_1b + p_2a)^2} \]

so that

\[ x_i \frac{\partial v}{\partial M} = -\frac{\partial v}{\partial p_i} \]

for goods \( i = 1, 2 \).

What level of expenditure would give the person the utility level \( u \). If she had \( E \) dollars to spend, then she would get utility of

\[ u = \frac{ab}{p_1b + p_2a} E \]

if the prices she faced were \( p_1 \) and \( p_2 \). That means that the expenditure \( E \) required to get the utility level \( u \) when prices are \( p_1 \) and \( p_2 \) is

\[ E(p_1, p_2, u) = \frac{p_1b + p_2a}{ab} u \]

which is the person’s expenditure function. The person’s Hicksian, or compensated, demands for the goods are the partial derivatives of the expenditure function with respect to the prices:

\[ E_1(p_1, p_2, u) = \frac{b}{ab} u \]

\[ E_2(p_1, p_2, u) = \frac{a}{ab} u \]

Notice that here the compensated demands for each good do not depend on the price of the good: with \( L \)-shaped indifference curves, the person will always locate at the kink on the indifference curve, regardless of prices.
Perfect Substitutes

If \( u(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n \)

then if \( \frac{a_1}{p_1} > \frac{a_2}{p_2} > \cdots > \frac{a_n}{p_n} \)

the person will consume only good 1:

\[
x_1 = \frac{M}{p_1}
\]

and \( x_i = 0 \) for every \( i > 1 \). Her level of utility would then be \( \frac{a_1M}{p_1} \). More generally, she consumes only the good(s) for which \( \frac{a_i}{p_i} \) is at a maximum. So

\[
v(p_1, p_2, \ldots, p_n, M) = \max \left( \frac{a_1M}{p_1}, \frac{a_2M}{p_2}, \ldots, \frac{a_nM}{p_n} \right)
\]

[ left to the reader : Roy’s Identity holds here. ] How much money \( E \) would she need to get the utility level \( u \)? She would spend all her money on the good \( i \) for which \( \frac{a_i}{p_i} \) is lowest. And to get utility \( u \) from consuming good \( i \) means consuming \( \frac{u}{a_i} \) units of the good, which will cost \( p_i \frac{u}{a_i} \). That means that

\[
E(p_1, p_2, \cdots, p_n, u) = \min \left( \frac{p_1u}{a_1}, \frac{p_2u}{a_2}, \cdots, \frac{p_nu}{a_n} \right)
\]

which makes the person’s compensated demand for good \( i \) \( \frac{u}{a_i} \) if good \( i \) yields the highest level of \( \frac{a_i}{p_i} \), and 0 otherwise.

Quasi–Linear Preferences

\[ iiia \quad u(x_1, x_2, x_3) = x_1 + 2\sqrt{x_2} + \ln x_3 \]

In this case, the demand functions are

\[
x_2 = (\frac{p_1}{p_2})^2 \quad (2'')
\]

\[
x_3 = \frac{p_1}{p_3} \quad (3'')
\]

\[
x_1 = \frac{M}{p_1} - \frac{p_1}{p_2} - 1
\]

if the person’s income \( M \) is high enough so that \( M > (p_1)^2/p_2 - p_1 \).

Substituting into the direct utility function, the person’s utility is

\[
[\frac{M}{p_1} - \frac{p_1}{p_2} - 1] + 2\sqrt{\left(\frac{p_1}{p_2}\right)^2} + \ln \left(\frac{p_1}{p_3}\right)
\]

so that

\[
v(p_1, p_2, p_3, M) = \frac{M}{p_1} + \frac{p_1}{p_2} + \ln p_1 - \ln p_3 - 1
\]

The expenditure function \( E(p_1, p_2, p_3, u) \) must ( always ) satisfy the condition

\[
v(p_1, p_2, p_3, E[p_1, p_2, p_3, u]) = u
\]

so that here

\[
u = \frac{E(p_1, p_2, p_3, u)}{p_1} + \frac{p_1}{p_2} + \ln p_1 - \ln p_3 - 1
\]
or

\[ E(p_1, p_2, p_3, u) = p_1 u - \frac{(p_1)^2}{p_2} - p_1 \ln p_1 + p_1 \ln p_3 + p_1 \]

which means that the compensated (Hicksian) demand functions are

\[ E_1(p_1, p_2, p_3, u) = u + 1 - 2\frac{p_1}{p_2} - \ln p_1 + \ln p_2 \]

\[ E_2(p_1, p_2, p_3, u) = \left(\frac{p_1}{p_2}\right)^2 \]

\[ E_3(p_1, p_2, p_3, u) = \frac{p_1}{p_3} \]

In this case, the Marshallian demand functions for goods 2 and 3 are the same as the Hicksian demand functions. This will always be the case with quasi-linear preferences: if the income elasticity of demand for a good is 0, then the Slutsky equation says that the compensated and uncompensated demand functions will be the same.

\[ iiiib \quad u(x_1, x_2, x_3) = x_1 + \ln x_2 + \ln x_2 + x_3 \]

Here the Marshallian demand functions are

\[ x_2 = \frac{p_1}{p_2 - p_3} \quad (2'') \]

\[ x_3 = \frac{p_1(p_2 - 2p_3)}{p_3(p_2 - p_3)} \quad (3'') \]

\[ x_1 = \frac{M}{p_1} - 2 \]

when \( M > 2p_1 \) and when \( p_2 > 2p_3 \).

Substituting in the definition of the direct utility function, and simplifying,

\[ v(p_1, p_2, p_3, M) = \frac{M}{p_1} + 2 \ln p_1 - \ln (p_2 - p_3) - \ln p_3 - 2 \]

This equation then implies that

\[ E(p_1, p_2, p_3, u) = p_1 u + 2p_1 - 2p_1 \ln p_1 + p_1 \ln (p_2 - p_3) + p_1 \ln p_3 \]

giving Hicksian demand functions

\[ E_1(p_1, p_2, p_3, u) = u - 2 \ln p_1 + \ln (p_2 - p_3) + \ln p_3 \]

\[ E_2(p_1, p_2, p_3, u) = \frac{p_1}{p_2 - p_3} \]

\[ E_3(p_1, p_2, p_3, u) = \frac{p_1}{p_3} - \frac{p_1}{p_2 - p_3} = \frac{p_1(p_2 - 2p_3)}{p_3(p_2 - p_3)} \]

so that again the Marshallian demand functions for goods 2 and 3 are the same as the Hicksian demand functions.

\[ iv \quad \text{Cobb–Douglas Preferences} \]

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\[ u(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \]

then

\[ x_i = \frac{a_i M}{a_1 + a_2 + \cdots + a_n} \quad i = 1, 2, \ldots, n \]

is the Marshallian demand function for good \( i \). That means that

\[ v(p_1, p_2, \ldots, p_n, M) = A^{a_1} p_2^{-a_2} \cdots p_n^{-a_n} M^a \]

where

\[ A \equiv (a_1)^{a_1} (a_2)^{a_2} \cdots (a_n)^{a_n} \]

and

\[ a = a_1 + a_2 + \cdots + a_n \]

( Note that the particular representation of the direct utility function matters for the indirect utility function. Taking a monotonically increasing transformation of \( u(x) \), such as taking \( U(x) = \ln(u(x)) \), will also mean taking that transformation of the indirect utility function, getting \( V(p, M) = \ln[ v(p, M) ] \) for example.)

The expenditure function in this case is

\[ E(p_1, p_2, \ldots, p_n, u) = B^{b_1} (p_2)^{b_2} \cdots (p_n)^{b_n} u^{1/A} \]

where

\[ B \equiv A^{1/a} \]

and

\[ b_i \equiv \frac{a_i}{a} \]

The Hicksian demand functions are

\[ E_i(p_1, p_2, \ldots, p_n, u) = b_i (p_1)^{a_1} (p_2)^{a_2} \cdots (p_{i-1})^{a_{i-1}} (p_i)^{a_i-1} (p_{i+1})^{a_{i+1}} \cdots (p_n)^{a_n} u^{1/A} \]

which shows that, when preferences are Cobb–Douglas, goods are net substitutes even though the Marshallian demand for good \( i \) does not depend on any other prices \( j \).

\( v \) CES Preferences

Done in the textbook.

\( vi \) Stone–Geary Preferences

\[ U(x) = b_1 \ln(x_1 - s_1) + b_2 \ln(x_2 - s_2) + \cdots + b_n \ln(x_n - s_n) \]

with

\[ b_1 + b_2 + \cdots + b_n = 1 \]

The Marshallian demands are

\[ x_i = s_i + \frac{b_i M}{p_i} \sum_{j=1}^{n} p_j s_j \]

which gives rise to an indirect utility function

\[ v(p_1, \ldots, p_n, M) = \ln(M - p \cdot s) + \sum_{i=1}^{n} b_i \ln p_i - \sum_{i=1}^{n} b_i \ln p_i \]
(where $s$ is the vector $(s_1, s_2, \ldots, s_n)$. Taking the exponents of both sides

$$e^u = (M - p \cdot s) \tilde{B}(p_1)^{b_1} (p_2)^{-b_2} \cdots (p_n)^{b_n}$$

where

$$\tilde{B} \equiv (b_1)^{b_1} (b_2)^{b_2} \cdots (b_n)^{b_n}$$

and where I have used the facts that $e^{a+b} = e^a e^b$ and that

$$e^{a \ln b} = b^a$$

Therefore

$$E(p_1, p_2, \ldots, p_n, u) = (e^u) \frac{1}{B} (p_1)^{b_1} (p_2)^{b_2} \cdots (p_n)^{b_n} + p \cdot s$$

leading to Hicksian demand functions

$$E_i(p_1, p_2, \ldots, p_n, u) = b_i(e^u) \frac{1}{B} (p_1)^{b_i} (p_2)^{a_2} \cdots (p_{i-1})^{a_{i-1}} (p_i)^{a_i-1} (p_{i+1})^{a_{i+1}} \cdots (p_n)^{a_n} + s_i$$