Chap 2: Nash Equilibrium: Theory

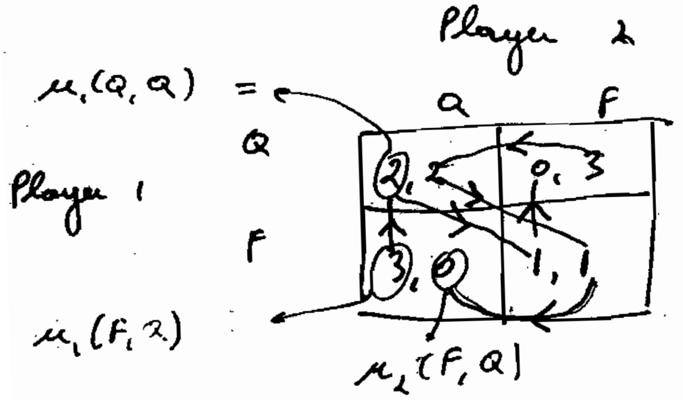
2.1 Strategic games

Definition 13.1 Strategic game with ordinal preferences:

- Set of Players P
- for each player, a set of Actions A_i, $i \in P$
- ———————, preferences over the set of action profiles $\succsim_i$ over $\prod_{i \in P} A_i$ $\hookrightarrow$ or u_i

Example: Prisoner's Dilemma

- $P = \{1, 2\}$
- $A_1 = A_2 = \{Q, F\}$
- $\succsim_1$ represented by payoff function u_1 s.t.
  $$u_1(F, Q) > u_1(Q, Q) > u_1(F, F) > u_1(Q, F)$$
- $\succsim_2$ represented by u_2 s.t.
  $$u_2(Q, F) > u_2(Q, Q) > u_2(F, F) > u_2(F, Q)$$



$\Longleftrightarrow$

Player 2

Player 1	Q	F
Q	10, 99	-6, 100
F	11, 97	-5, 98

examples of situations
with a Prisoner's Dilemma:

Working on a joint project:

	work hard	Goof off
work hard	2, 2	0, 3
Goof off	3, 0	1, 1

Exerc. 16.1

Duopoly:

	High	Low
High	1000, 1000	−200, 1200
Low	1200, −200	600, 600

Arms race:

	D	B
Don't build bombs	2, 2	0, 3
Build bombs	3, 0	1, 1

Common Property

	Little	Lot
Graze a little	2, 2	0, 3
Graze a lot	3, 0	1, 1

Exercise 17.1

	X	Y
X	3, 3	1, 5
Y	5, 1	0, 0

$u_1(Y,X) > u_1(X,X) > u_1(X,Y) > u_1(Y,Y)$

$\Rightarrow$ not prisoner's dilemma

Exercise 18.1

2.3 Example: BoS

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

2.4 Example: Matching Pennies

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

2.5 Example: Stag Hunt

Players: $P = \{\text{Hunter 1}, \text{Hunter 2}\}$

Actions: $A_1 = A_2 = \{\text{Stag}, \text{Hare}\}$

Preferences: $u_1(\text{Stag}, \text{Stag}) > u_1(\text{Hare}, \text{Stag}) = u_1(\text{Hare}, \text{Hare}) > u_1(\text{Stag}, \text{Hare})$

$u_2(\text{Stag}, \text{Stag}) > u_2(\text{Stag}, \text{Hare}) = u_2(\text{Hare}, \text{Hare}) > u_2(\text{Hare}, \text{Stag})$

	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

- extension to n hunters

"Security dilemma":

	D	B
Don't Build Bombs	3,3	0,2
Build Bombs	2,0	1,1

2.6 Nash Equilibrium

Definition 23.1 Nash Equilibrium of strategic game with ordinal preferences

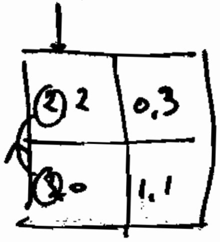
Let $G = (P, (A_i), (u_i))$ be a strategic game with ordinal preferences.
Let a^* be a strategy profile of G (that is, $a^* \in \prod_{i \in P} A_i$).
a^* is a Nash equilibrium of G iff $u_i(a^*) \geq u_i(a_i, a^*_{-i})$,
for all $a_i \in A_i$ and all $i \in P$.

2.7 Examples of N.E.

2.7.1 Prisoner's Dilemma:

	Q	F
Q	2,2	0,3
F	3,0	1,1

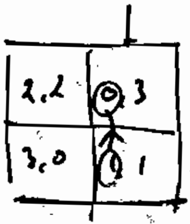
(Q, Q) :



2,2	0,3
3,0	1,1

→ $u_1(F,Q) > u_1(Q,Q) \Rightarrow (Q,Q)$ not NE.

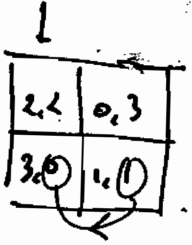
(Q, F):



2,2	0,3
3,0	1,1

→ $u_1(F,F) > u_1(Q,F) \Rightarrow (Q,F)$ not N.E.

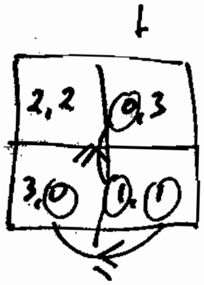
(F, Q) :



2,2	0,3
3,0	1,1

→ $u_2(F,F) > u_2(F,Q) \Rightarrow (F,Q)$ not N.E.

(F, F) :



2,2	0,3
3,0	1,1

→ $u_1(Q,F) \le u_1(F,F)$

$u_2(F,Q) \le u_2(F,F)$ $\Rightarrow (F,F)$ is N.E.

Exercise 27.1 : Variant of Prisoner's Dilemma with altruistic preferences

receipts of money: $(m_1(a), m_2(a))$

	Q	F
Q	2,2	0,3
F	3,0	1,1

$\alpha \geq 0$

players' payoffs : $u_i(a) = m_i(a) + \alpha m_j(a)$ (representing preferences) $j \neq i$

a. $\alpha = 1$

	Q	F
Q	4,4	3,3
F	3,3	2,2

not equivalent to Prisoner's Dilemma

6:

	Q	F
Q	$2(1+\alpha), 2(1+\alpha)$	$3\alpha, 3$
F	$3, 3\alpha$	$1+\alpha, 1+\alpha$

$\boxed{<}$ ⊜

equivalent to Prisoner's Dilemma iff: $3 > 2(1+\alpha) \iff \alpha < \frac{1}{2}$

⊜ $\boxed{<}$

$1+\alpha > 3\alpha \iff \alpha < \frac{1}{2}$

$2(1+\alpha) > 1+\alpha$ ← always true

↓

only N.E. $= (F, F)$

iff $\alpha < \frac{1}{2}$

If $\alpha = \frac{1}{2}$

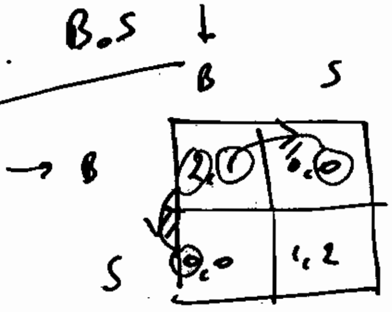
	Q	F
Q	3,3	$\frac{3}{2}$, 3
F	3, $\frac{3}{2}$	$\frac{3}{2}, \frac{3}{2}$

(Q,Q), (Q,F), (F,Q), (F,F) are N.E.

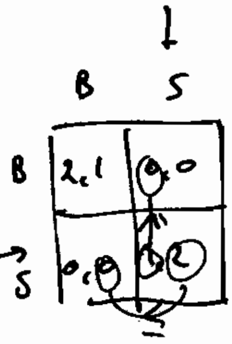
If $\alpha > \frac{1}{2}$: (Q, Q) is the only N.E.

Exercise [illegible]

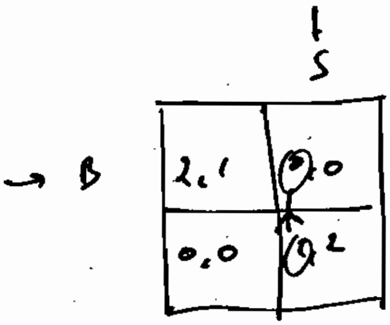
2.1.2. B.S



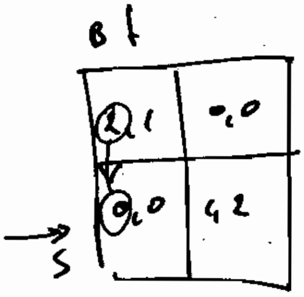
→ (B, B) NE.



→ (S, S) NE

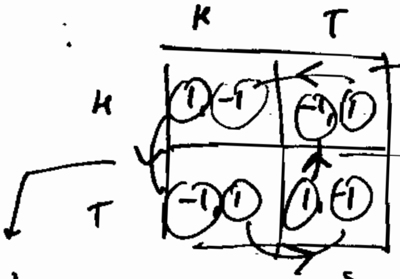


→(B,S) not NE



→ (S, B) not NE

2.7.3 Matching Pennies :



	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

→ (H, H) not N.E.

→ (H, T) not N.E. ⇒ no NE

→ (T, T) not N.E

H)
ot NE

.7.4. Stag hunt :

	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

2 N.E.: (Stag, Stag)
(Hare, Hare)

Stag hunt with n hunters :

<u>2 N.E</u>: $s^* = (Stag, Stag, \dots, Stag)$ because
$u_i(Hare, s^*_{-i}) < u_i(s^*)$ for all i

$h^* = (Hare, Hare, ---, Hare)$ because
$u_i(Stag, h^*_{-i}) < u_i(h^*)$ for all i

If a is a strategy profile $\neq s^*$ and $\neq h^*$

$$\Leftrightarrow \quad a = (\dots, \overset{i}{\overset{\swarrow}{Stag}}, \dots, Hare, \dots)$$

Then $u_i(Hare, a_{-i}) > u_i(a)$

$\Downarrow$

a is not NE.

$\boxed{\text{Exercise 30.1}}$

$\boxed{\text{Exercise 31.1}}$ G

2.7.5 **Hawk-Dove**

Ex. 31.2: $\overset{\overset{3}{\shortparallel}}{u_1}(H,D) > \overset{\overset{2}{\shortparallel}}{u_1}(D,D) > \overset{\overset{1}{\shortparallel}}{u_1}(D,H) > \overset{\overset{0}{\shortparallel}}{u_1}(H,H)$

$u_2(D,H) > u_2(D,D) > u_2(H,D) > u_2(H,H)$

	H	D
H	0,0	3,1
D	1,3	2,2

2 N.E.: (H, D), (D, H)

2.7.6: **Coordination game:**

	B	S
B	2,2	0,0
S	0,0	1,1

2 N.E.: (B, B), (S, S)

2.7.7: Provision of a public good

~~exercise~~ 33.1: Players: n people

$k \leq n$

Actions: $A_1 = \dots = A_n = \{$Contribute, Don't contribute$\}$

Preferences:

for strategy profile a: public good is provided iff at least k people contribute

$u_i(a) > u_i(a') > u_i(a'') > u_i(a''')$

$u_i(a)$: public good is provided, a_i = Don't contribute

$u_i(a')$: Public good is provided, a_i = Contribute

$u_i(a'')$: Public [illegible] is [illegible] a_i = [illegible]

$u_i(a''')$: [illegible] a_i = [illegible]

→ **N.E.:** a^* is a N.E. iff $a^* \equiv$ (Don't contribute, Don't contribute, ..., Don'

or

people who contribute $= k$

2.7.8 : Strict and nonstrict equilibria

Definition: Strict Nash Equilibrium

Let $G=(P,(A_i),(u_i))$ be a strategic game with ordinal preferences

Let a^* be a strategy profile of G

a^* is a strict Nash equilibrium of G iff $u_i(a^*) > u_i(a_i, a^*_{-i})$
$\left(\begin{array}{l} \text{for all } a_i \in A_i \text{ s.t. } a_i \neq a^*_i \\ \text{and for all } i \in P. \end{array} \right.$

$$a^* \text{ is a strict N.E. } \begin{array}{c} \Longrightarrow \\ \cancel{\Longleftarrow} \end{array} a^* \text{ is a N.E.}$$

Example:

	L	M	R
T	1,1	1,0	0,1
B	1,0	0,1	1,0

only NE = (T,L)
↓
not a strict N.E.

2.7.9 Additional examples:

Exercise 34.1 G

Exercise 34.2 G

Exercise 34.3 G

2.8 Best Response Functions

2.8.1

Definition : Best response function

Let $G = (P, (A_i), (u_i))$ be a strategic game with ordinal preferences

The best response function of player i is the function B_i s.t.

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}), \text{ for all } a_i' \text{ in } A_i\} \subseteq A_i,$$

$a_{-i} \in A_{-i}$

Example: B.S

	B	S
B	2,1	0,0
S	0,0	1,2

$B_1(B) = \{B\}$ $B_2(B) = \{B\}$

$B_1(S) = \{S\}$ $B_2(S) = \{S\}$

2.8.2

Proposition 36.1 : Let $G = (P, (A_i), (u_i))$ be a game and let a^* be a strategy profile of G. Then,

a^* is a N.E. of G iff $a_i^* \in B_i(a_{-i}^*)$, for all $i \in P$.

2.8.3 Using best response functions to find N.E.

Example :

	L	C	R
T	$1, 2^*$	$2^*, 1$	$1^*, 0$
M	$2^*, 1^*$	$0, 1^*$	$0, 0$
B	$0, 1$	$0, 0$	$1^*, 2^*$

$_2(T) = L$

$_1(L) = \{M\}$

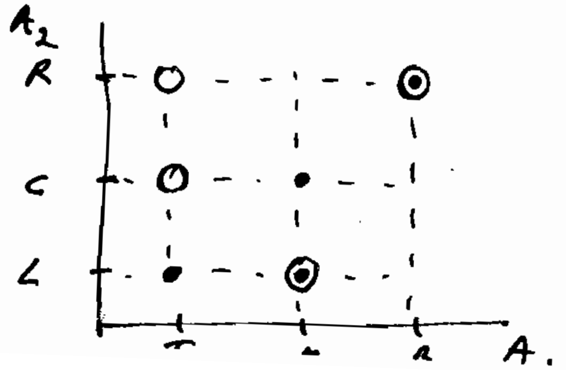
$B_1(C) = T$

$B_1(R) = \{T, B\}$

$B_2(B) = \{R\}$

$B_2(M) = \{L, C\}$

Other representation:



2 N.E. :

(M, L) because $\begin{cases} M \in B_1(L) \\ L \in B_2(M) \end{cases}$

(B, R) because $\begin{cases} B \in B_1(R) \\ R \in B_2(B) \end{cases}$

$\boxed{\text{Exercise 37.1}}$

$\boxed{\text{Exercise 38.1}}$

$\boxed{\text{Exercise 38.2}}$

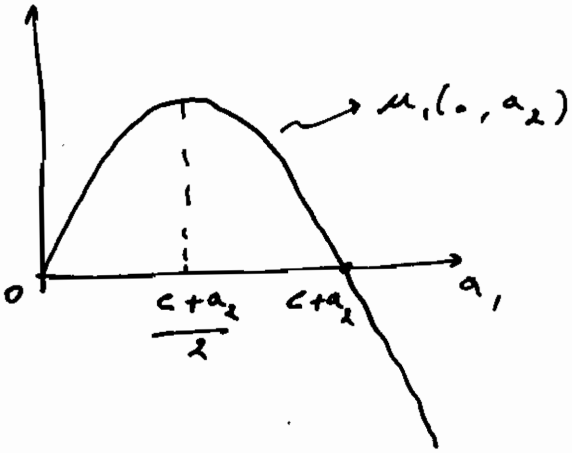
Example 39.1 (a synergistic relationship)

Players: $P = \{1, 2\}$

Actions: $A_1 = A_2 = [0, +\infty)$

Preferences: $u_1(a_1, a_2) = a_1(c + a_2 - a_1)$ where $c > 0$

$u_2(a_1, a_2) = a_2(c + a_1 - a_2)$

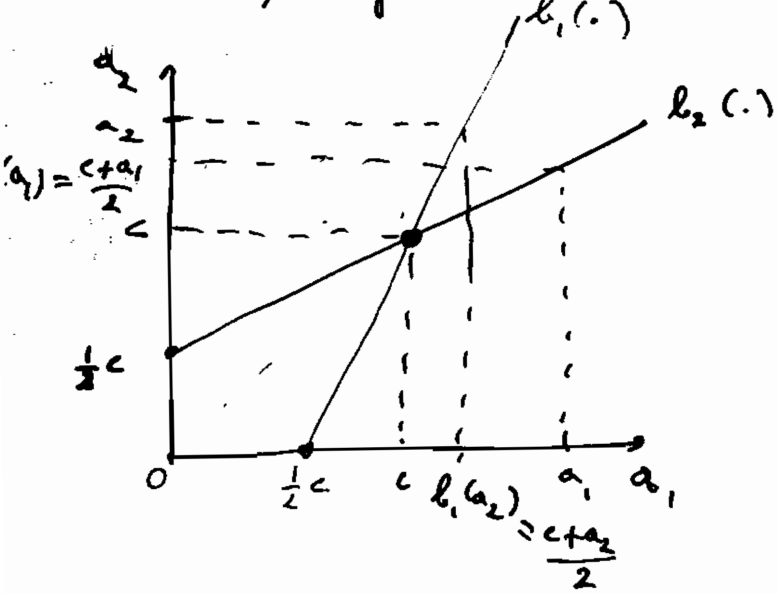


$\Rightarrow B_1(a_2) = \{b_1(a_2)\}$ where $b_1(a_2) = \frac{c+a_2}{2}$

Similarly:

$B_2(a_1) = \{b_2(a_1)\}$ where $b_2(a_1) = \frac{c+a_1}{2}$

Best response functions:



N.E.: (a_1^*, a_2^*) solution of

$$\begin{cases} a_1^* = b_1(a_2^*) \\ a_2^* = b_2(a_1^*) \end{cases}$$

$$\begin{cases} a_1^* = \frac{1}{2}(c + a_2^*) \\ a_2^* = \frac{1}{2}(c + a_1^*) \end{cases}$$

$\hookrightarrow (a_1^*, a_2^*) = (c, c)$

$\boxed{\text{Exercise } 41.1}$

$\boxed{\text{Exercise } 42.1}$

$\boxed{\text{Exercise } 42.2}$

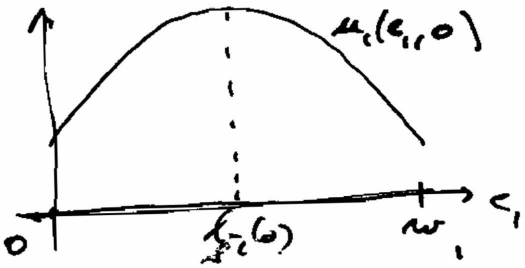
2.8.4 : Illustration : contributing to a public good

Players: $P = \{1, 2\}$

Actions: $A_i := [0, w_i]$ ⟶ player i's wealth

Preferences: $u_i(c_1, c_2) = v_i(c_1 + c_2) (+ w_i) - c_i$, $i = 1, 2$

Assume the graph of $u_1(c_1, 0) = v_1(c_1 + 0) - c_1$ is



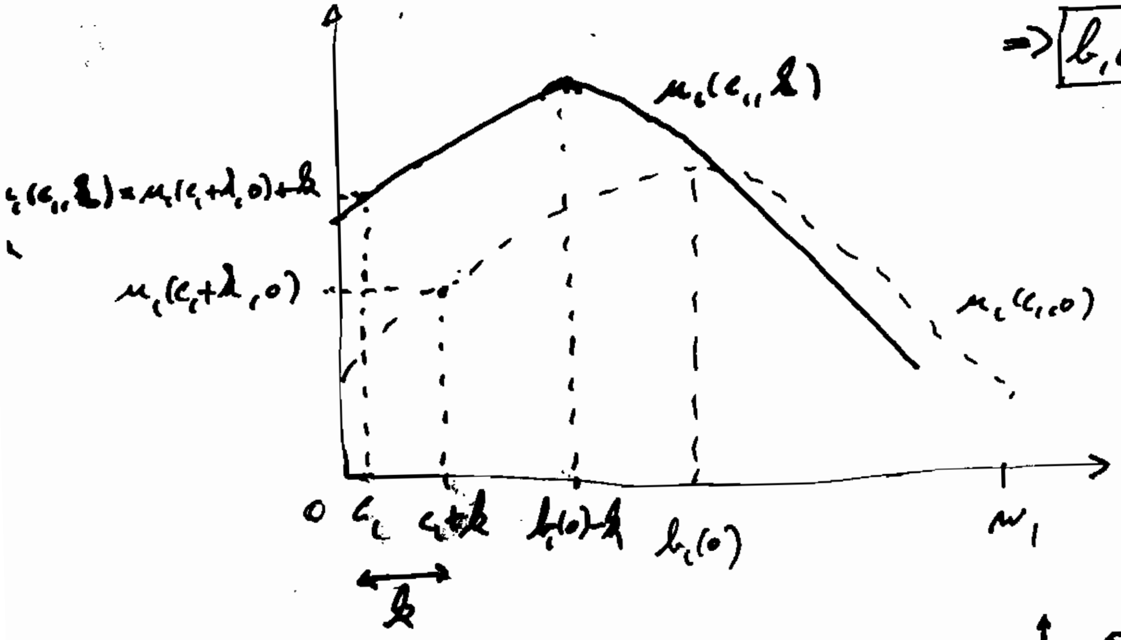
Then : $B_1(0) = \{b_1(0)\}$

$b_1(0)$ solution of $\underset{c_1}{\text{Max}}(\underbrace{v_1(c_1+0)-c_1}_{=u_1(c_1,0)})$ $0 < b_1(0) < w_1$

If $c_2 = k \geq 0$.

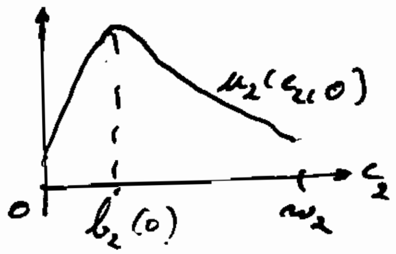
$$\begin{aligned} u_1(c_1, k) &= v_1(c_1 + k) - c_1 \\ &= v_1(c_1 + k) - (c_1 + k) + k \\ &= u_1(c_1 + k, 0) + k \end{aligned}$$

$\Rightarrow$



$\Rightarrow \boxed{b_1(k) = \max(b_1(0) - k, 0)}$

ssume the graph of $u_2(0, c_2) = v_2(0 + c_2) - c_2$ is

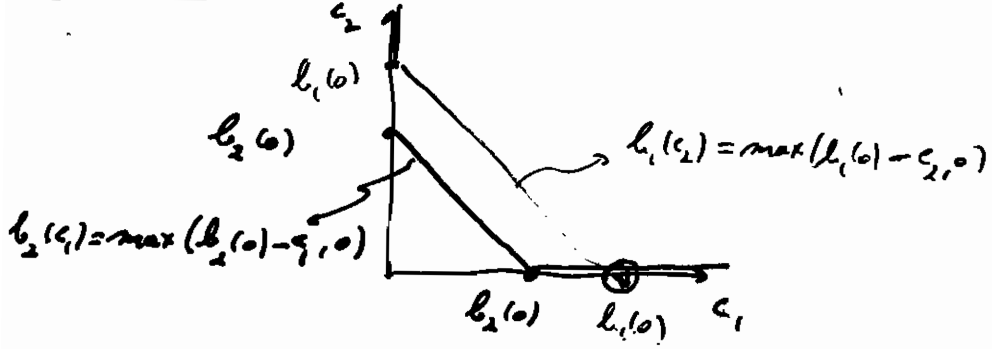


n :

$\boxed{b_2(k) = \max(b_2(0) - k, 0)}$

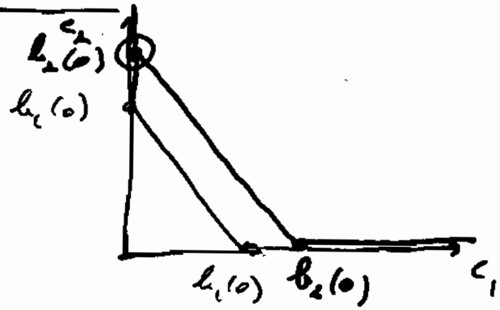
Case 1 : $\ell_1(0) > b_2(0)$

Set of Nash Equilibria



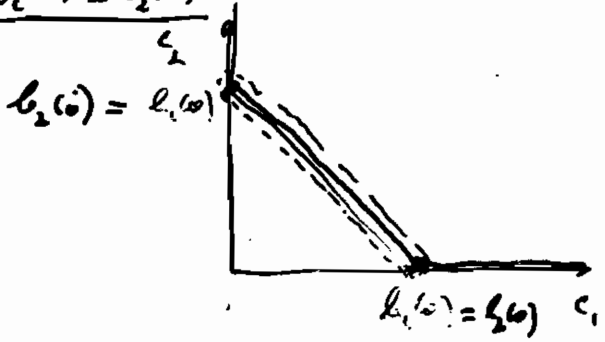
$\{(\ell_1(0), 0)\}$

ase 2 : $\ell_1(0) < b_2(0)$:



$\{(0, b_2(0))\}$

se 3 : $\ell_1(0) = \ell_2(0)$



$\{(c_1, c_2) \mid c_1 + c_2 = \ell_1(0) \text{ and } c_1, c_2 \geq 0\}$

$\boxed{\text{Exercise } 44.1}$

.9 Dominated actions

2.9.1

Definition 45.1 : Strict domination

et $G = (P, (A_i), (A_i))$ be a strategic game with ordinal preferences

t a_i', a_i'' be actions available to player i, that is, $a_i', a_i'' \in A_i$.

$_i''$ strictly dominates a_i' iff $u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$,

for all $a_{-i} \in A_{-i}$

$_i'$ is strictly dominated iff there exists $a_i \in A_i$ s.t.

a_i strictly dominates a_i'

Consequence of the definitions :

a strictly dominated action is not used in any Nash equilibrium

2.9.2

Definition 46.1 : Weak domination

Let $G = (P, (A_i), (\succsim_i))$ be a strategic game with ordinal preferences
let a_i', a_i'' be in A_i.

a_i'' weakly dominates a_i' iff $u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i})$. for all $a_{-i} \in A_{-i}$
and
there exists $a_{-i} \in A_{-i}$ s.t. $u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$.

a_i' is weakly dominated iff there exists $a_i'' \in A_i$ s.t. a_i'' weakly dominates a_i'.

Consequence of the definitions :

a weakly dominated action is not used in any strict Nash equilibrium

example:

	B	C
B	1,1	2,0
C	0,2	2,2

Player 1 : C is weakly dominated

Player 2 : ____________

Notice that (C, C) is a N.E.

Exercise 47.1

Exercise 47.2

~~2.9.3~~

~~2.9.4~~

2.10 Equilibrium in a single population: symmetric games and symmetric equilibria

Definition 51.1: Symmetric two-player strategic game with ordinal preferences
Let $G = (P = \{1,2\}, (A_i), (u_i))$ be a two-player strategic game with ordinal preferences.

$$G \text{ is symmetric } \underline{\text{iff}} \begin{cases} A_1 = A_2 \\ \text{and} \\ u_1(a_1, a_2) = u_2(a_2, a_1), \text{ for all } (a_1, a_2) \in A_1 \times A_2 \end{cases}$$

Example: Symmetric two-player strategic game with two actions

	A	B
A	w, w	x, y
B	y, x	z, z

Exercise 51.2

Definition 52.1: Symmetric Nash equilibrium
Let $G = (P = \{1, 2, \dots, n\}, (A_i), (u_i))$ be a strategic game s.t. $A_1 = \dots = A_n$.
Let a^* be a strategy profile of G.

$$a^* \text{ is a symmetric N.E. } \underline{\text{iff}} \begin{cases} a^* \text{ is a N.E.} \\ \text{and} \\ a^*_i = a^*_j \text{, for all } i, j = 1, \dots, n \end{cases}$$

Examples:

	L	R
L	1, 1	0, 0
R	0, 0	1, 1

Symmetric game
- 2 N.E.: (L, L), (R, R)
- 2 symmetric N.E.: (L, L), (R, R)

	X	Y
X	0, 0	1, 1
Y	1, 1	0, 0

Symmetric game
- 2 N.E.: (X, Y), (Y, X)
- 0 symmetric N.E.: /

Exercise 53.2

Chap 3:

NASH EQUILIBRIUM: ILLUSTRATIONS

3.1 Cournot's model of oligopoly

3.1.2 General model:

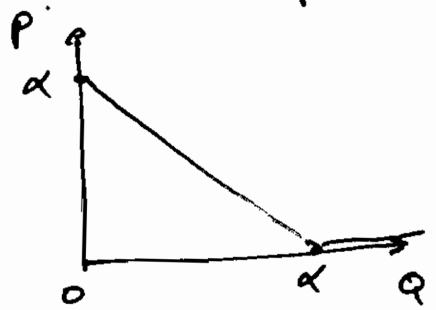
Players: $P = \{1, \dots, n\}$

Actions: $A_1 = \dots = A_n = [0, +\infty)$

Preferences: $$\pi_i(q_1, \dots, q_n) = q_i \underbrace{P(q_1 + \dots + q_n)}_{\text{inverse demand function}} - \underbrace{C_i(q_i)}_{\text{cost function}}$$

3.13 Example: duopoly with constant unit cost and linear inverse demand function

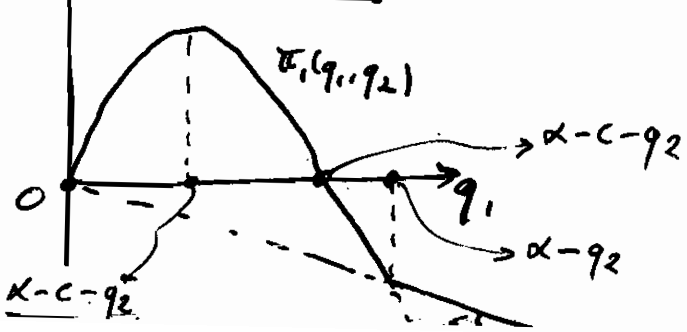
$n = 2$; $C_i(q_i) = c \cdot q_i$, $c \geq 0$; $P(Q) = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases}$, where $\alpha > c$



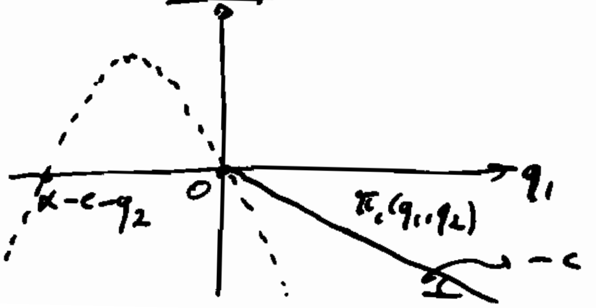
Assume $\alpha > c$

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - c \cdot q_1 = \begin{cases} q_1(\alpha - c - q_1 - q_2), & \text{if } q_1 + q_2 \leq \alpha \\ -cq_1, & \text{if } q_1 + q_2 > \alpha \end{cases}$$

If $q_2 \leq \alpha - c$:



If $q_2 > \alpha - c$:



3.2 Bertrand's model of oligopoly

3.2.1 General model:

<u>Players</u>: $P = \{1, \dots, n\}$

<u>Actions</u>: $A_1 = \dots = A_n = [0, +\infty)$

<u>Preferences</u>: $\pi_i(p_1, \dots, p_n) = p_i \frac{D(p_i)}{m} - C_i\left(\frac{D(p_i)}{m}\right)$, if $p_i = \min(p_1, \dots, p_n)$

, demand function

: cost function

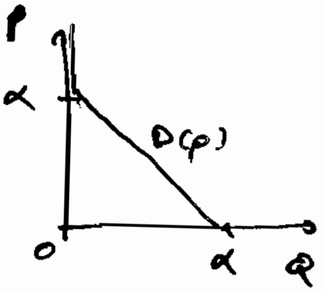
where $m = \#\{j \text{ s.t. } p_j = \min(p_1, \dots, p_n)\}$

$= 0$, if $p_i > \min(p_1, \dots, p_n)$

3.2.2 Example: duopoly with constant unit and linear demand functions

$n = 2$; $C_i(q_i) = c\, q_i$, $c \geq 0$; $D(p) = \alpha - p$ if $p \leq \alpha$

$= 0$ if $p > \alpha$



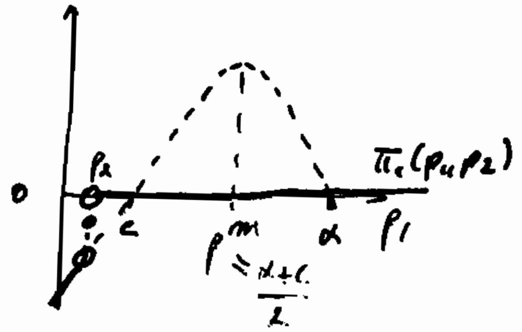
Assume $\alpha > c$

$$\pi_1(p_1, p_2) = p_1 D(p_1) - c\, D(p_1) = (p_1 - c)(\alpha - p_1), \text{ if } p_1 < p_2$$

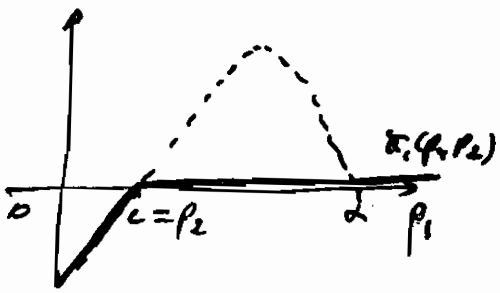
$$= p_1 \frac{D(p_1)}{2} - c \frac{D(p_1)}{2} = \frac{1}{2}(p_1 - c)(\alpha - p_1), \text{ if } p_1 = p_2$$

$$= 0, \text{ if } p_1 > p_2$$

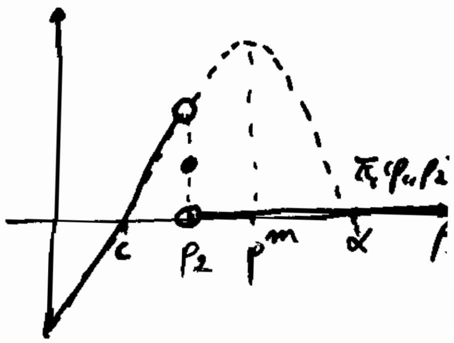
If $p_2 < c$:



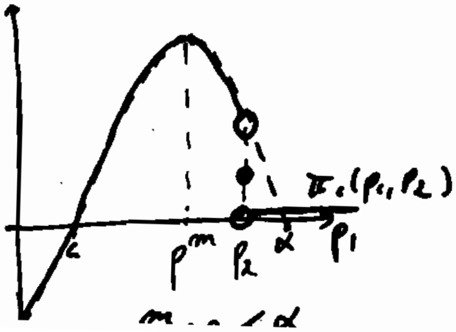
If $p_2 = c$:

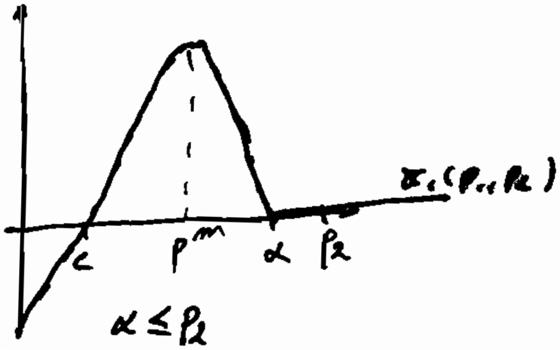


If $c < p_2 \leq p^m$:



If $p_2 > p^m$:

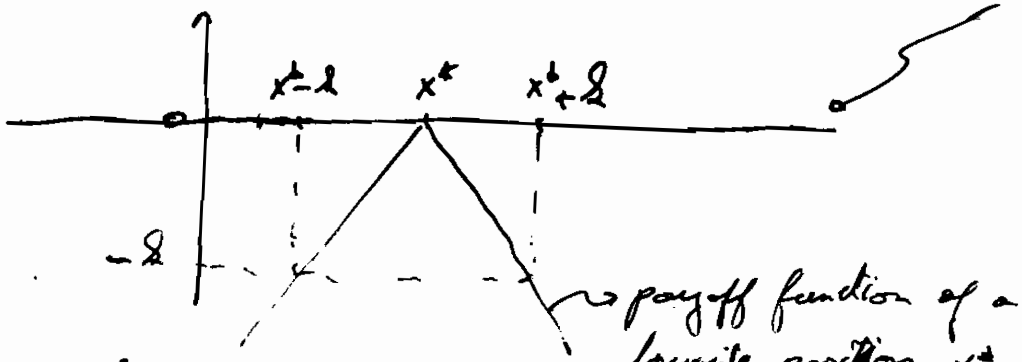




$\alpha \leq p_2$

3.3 Electoral competition

Voters' favorite positions are distributed over the real line



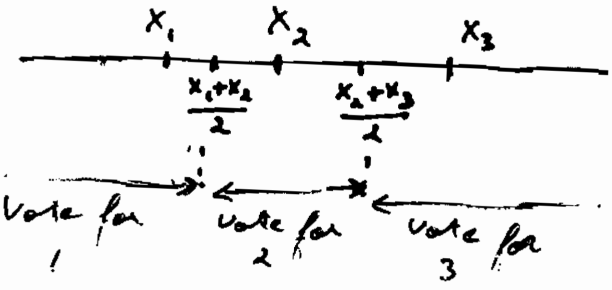
→ payoff function of a voter with favorite position x^*

Let m be the median of the favorite positions, that is: $F(m)=\frac{1}{2}$

cumulative distribution function of the favorite positions

Every voter votes for the position he prefers

Example: Position of candidate i $= x_i$, $i=1,2,3$



Proportion of the electorate going to candidate 1 $= F(\frac{x_1+x_2}{2})$

—— 2 $= F(\frac{x_2+x_3}{2}) - F(\frac{x_1+x_2}{2})$

—— 3 $= 1 - F(\frac{x_2+x_3}{2})$

If, for example, $F(\frac{x_1+x_2}{2}) > F(\frac{x_2+x_3}{2}) - F(\frac{x_1+x_2}{2})$ and $> 1 - F(\frac{x_2+x_3}{2})$, then candidate 1 wins

Hotelling's model of electoral competition (between candidates)

Players: $I = \{1,\dots,n\}$

Actions: $A_1 = \dots = A_n = (-\infty,+\infty)$

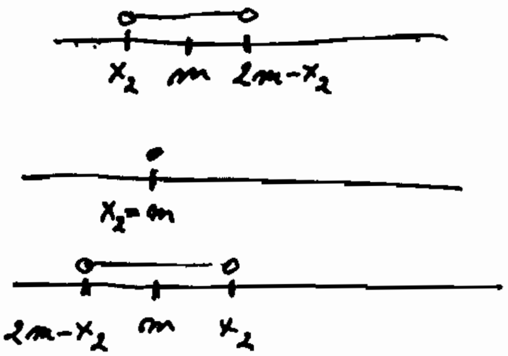
Preferences:

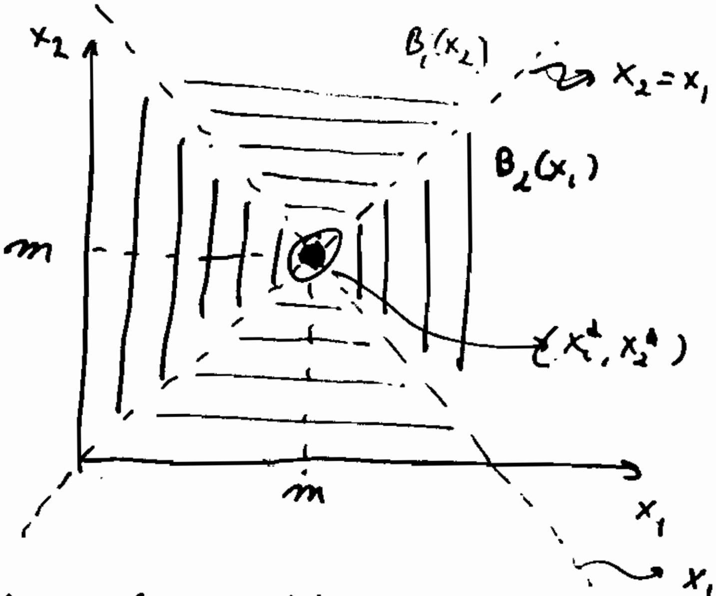
$$u_i(x_1,\dots,x_n) = \begin{cases} \xrightarrow{\text{if candidate } i \text{ wins}} n - \#\{\text{candidate } j \mid j\neq i,\ j \text{ ties with } i\} \\ \xrightarrow[\text{if candidate } i \text{ loses}]{} 0 \end{cases}$$

Case of n=2 candidates

Best response functions :

$$B_1(x_2) = \begin{cases} (x_2, 2m-x_2), & \text{if } x_2 < m \\ \{m\}, & \text{if } x_2 = m \\ (2m-x_2, x_2), & \text{if } x_2 > m \end{cases}$$





$$B_2(x_1) = \begin{cases} (x_1, 2m-x_1), & \text{if } x_1 < m \\ \{m\}, & \text{if } x_1 = m \\ (2m-x_1, x_1), & \text{if } x_1 > m \end{cases}$$

Nash Equilibrium : (x_1^*, x_2^*) solution of

$$\begin{cases} x_1^* \in B_1(x_2^*) \\ x_2^* \in B_2(x_1^*) \end{cases} \longrightarrow (x_1^*, x_2^*) = (m, m)$$

$\boxed{\text{Exercise } 73.1}$

$\boxed{\text{Exercise } 74.1}$

$\boxed{\text{Exercise } 74.2}$ G

$\boxed{\text{Exercise } 75.1}$ G

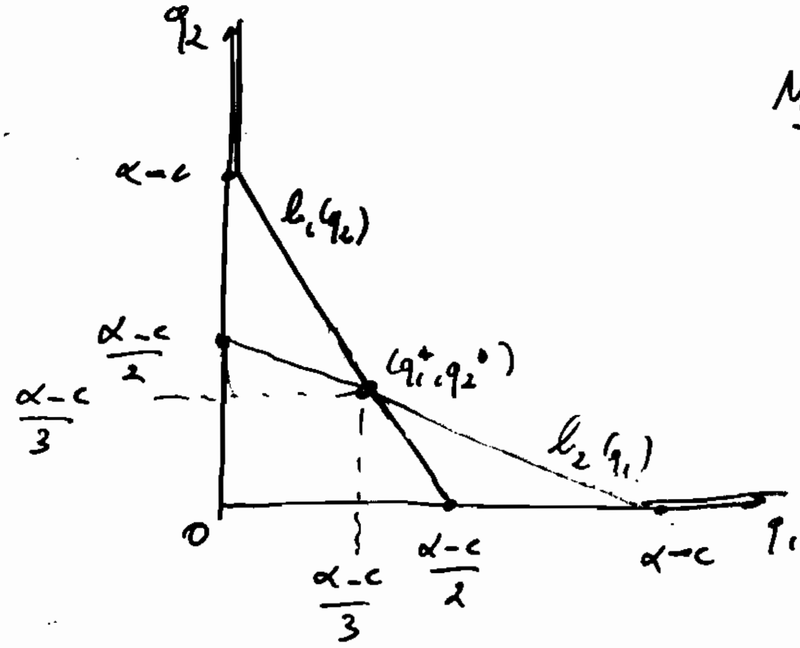
$\boxed{\text{Exercise } 75.2}$ G

$(\boxed{\text{Exercise } 75.3})$

$\boxed{\text{Exercise } 76.1}$

Best response functions: $B_1(q_2)=\{b_1(q_2)\}$, $B_2(q_1)=\{b_2(q_1)\}$ with

$$b_1(q_2)=\begin{cases}\frac{\alpha-c-q_2}{2} & \text{if } q_2\le\alpha-c\\ =0 & \text{if } q_2>\alpha-c\end{cases}\qquad b_2(q_1)=\begin{cases}\frac{\alpha-c-q_1}{2} & \text{if } q_1\le\alpha-c\\ =0 & \text{if } q_1>\alpha-c\end{cases}$$



Nash Equilibrium: (q_1^*,q_2^*) solution of

$$\begin{cases}q_1^*=b_1(q_2^*)\\ q_2^*=b_2(q_1^*)\end{cases}$$

$$\Downarrow$$

$$\begin{cases}q_1^*=\frac{\alpha-c-q_2^*}{2}\\ q_2^*=\frac{\alpha-c-q_1^*}{2}\end{cases}$$

$$\Downarrow$$

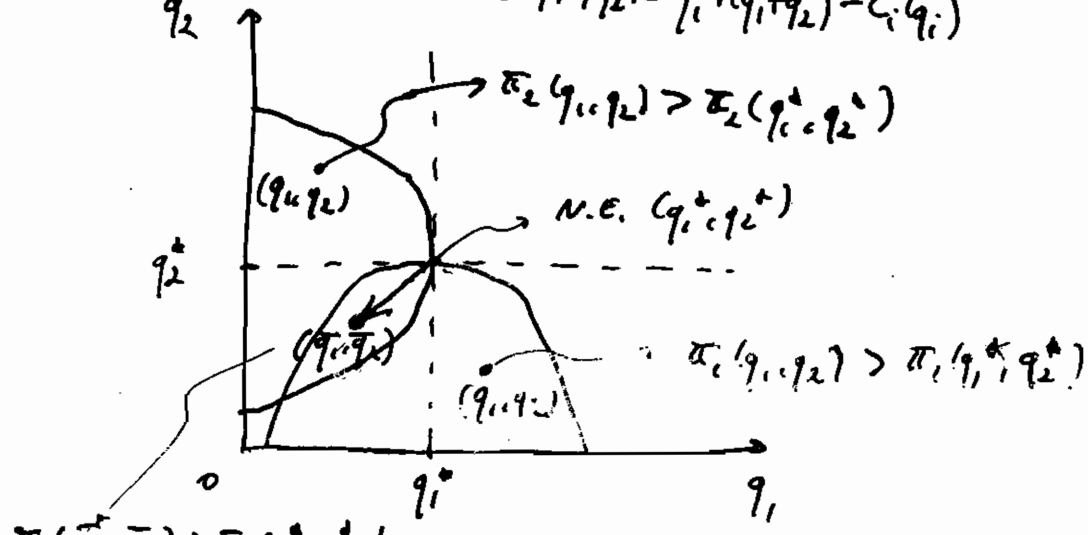
$$(q_1^*,q_2^*)=\left(\frac{\alpha-c}{3},\frac{\alpha-c}{3}\right)$$

$q_1^*+q_2^*=\frac{2}{3}(\alpha-c)$

$P(q_1^*+q_2^*)=\frac{1}{3}(\alpha+2c)$

3.1.4 Properties of N.E.:

$\pi_i(q_1,q_2)=q_i P(q_1+q_2)-C_i(q_i)$



$\pi_1(\bar q_1,\bar q_2)>\pi_1(q_1^*,q_2^*)$ /

$\pi_2(\bar q_1,\bar q_2)>\pi_2(q_1^*,q_2^*)$

Exercise 58.1

Exercise 59.1

Exercise 59.2

Exercise 60.1

Exercise 60.2

Exercise 61.1

Exercise 62.1 G

More generally such a $(\bar q_1,\bar q_2)$ → smooth exists when $\pi_i(q_1,q_2)=f_i(q_i,\underline{q_1+q_2})$

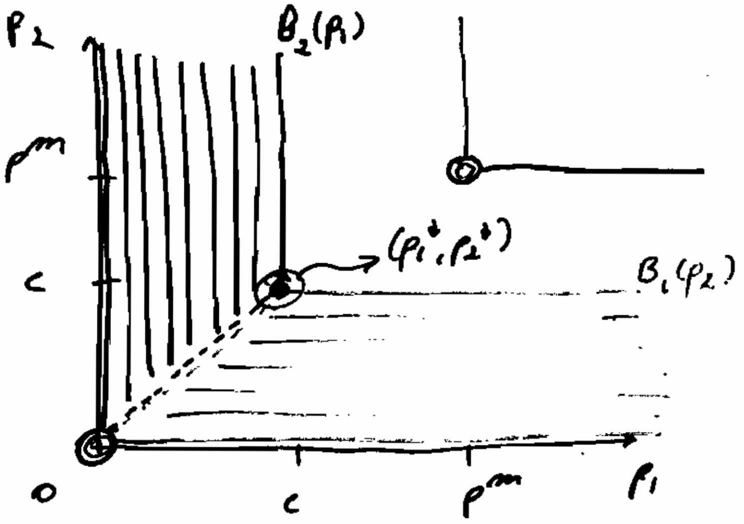
(and f_i str. decreasing w.r.t. ②)

Exercise 63.1

Best response functions:

$$B_1(p_2) = \begin{cases} (p_2, +\infty), & \text{if } p_2 < c \\ = [p_2, +\infty), & \text{if } p_2 = c \\ = \phi, & \text{if } c < p_2 \leq p^m \\ = \{p^m\}, & \text{if } p_2 > p^m \end{cases}$$

$$B_2(p_1) = \begin{cases} (p_1, +\infty), & \text{if } p_1 < c \\ = [p_1, +\infty), & \text{if } p_1 = c \\ = \phi, & \text{if } c < p_1 \leq p^m \\ = \{p^m\}, & \text{if } p_1 > p^m \end{cases}$$



Nash Equilibrium: (p_1^*, p_2^*) solution of

$$\begin{cases} p_1^* \in B_1(p_2^*) \\ p_2^* \in B_2(p_1^*) \end{cases}$$

$$\downarrow$$

$$(p_1^*, p_2^*) = (c, c)$$

Exercise 67.1

Exercise 67.2

Exercise 68.1

Exercise 68.2

Exercise 69.1 G

3.2-3 Discussions

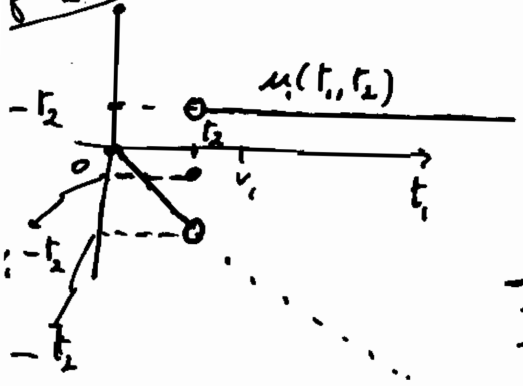
3.4 The war of attrition

Players: $I=\{1,2\}$ $0<v_i=$ player i's valuation

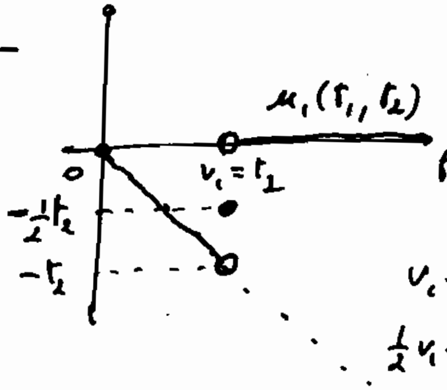
Actions: $A_1=A_2=[0,+\infty)$

Preferences:
$$u_i(t_1,t_2)=\begin{cases}-t_i\,, & \text{if } t_i<t_j\\ \frac{1}{2}v_i-t_i\,, & \text{if } t_i=t_j\\ v_i-t_j\,, & \text{if } t_i>t_j\end{cases}\qquad i\neq j$$

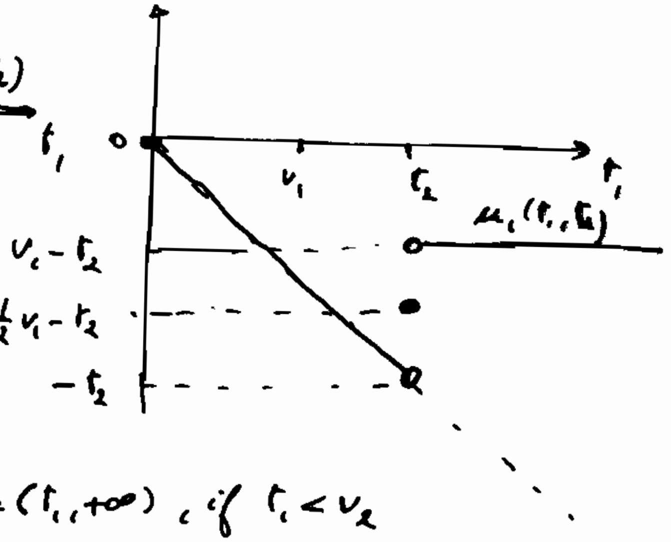
If $t_2<v_1$:



If $t_2=v_1$:



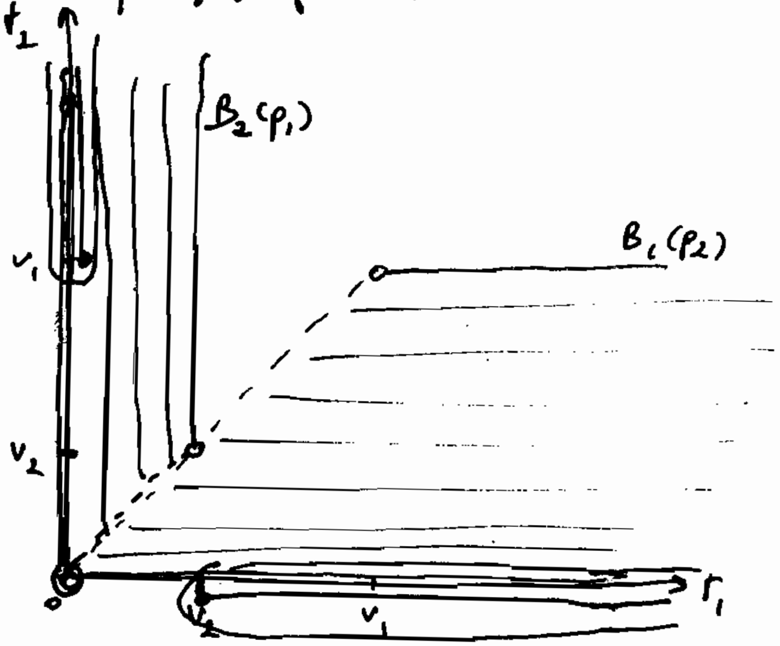
If $t_2>v_1$:



Best response functions:

$$B_1(t_2)=\begin{cases}(t_2,+\infty)\,, & \text{if } t_2<v_1\\ \{0\}\cup(t_2,+\infty)\,, & \text{if } t_2=v_1\\ \{0\}\,, & \text{if } t_2>v_1\end{cases}$$

$$B_2(t_1)=\begin{cases}(t_1,+\infty)\,, & \text{if } t_1<v_2\\ \{0\}\cup(t_1,+\infty)\,, & \text{if } t_1=v_2\\ \{0\}\,, & \text{if } t_1>v_2\end{cases}$$



Set of Nash Equilibria:

$$\{(0,t_2)\mid t_2\geqslant v_1\}\cup\{(t_1,0)\mid t_1\geqslant v_2\}.$$

$\boxed{\text{Exercise 79.1}}$

$\boxed{\text{Exercise 79.2}}$

$\boxed{\text{Exercise 80.1}}$

$\boxed{\text{Exercise 80.2}}$

3.5 Auctions

3.5.2 Second-price sealed-bid auction

Players: $P = \{1, \dots, n\}$

Actions: $A_1 = \dots = A_n = [0, +\infty)$

[Bidder i's valuation $= v_i > 0$
Assume: $v_1 > v_2 > \dots > v_n$

Preferences:

$$u_i(b_1, \dots, b_n) = \begin{cases} v_i - \max\limits_{j \neq i} b_j, & \text{if } \left\{ \begin{array}{l} b_i = \max(b_1, \dots, b_n) \\ \quad \text{and} \\ i \leq k, \text{ for all } k \text{ s.t.} \\ \qquad b_k = \max(b_1, \dots, b_n) \end{array} \right. \\ = 0, & \text{otherwise} \end{cases}$$

Examples of N.E.:

$(v_1, v_2, \dots, v_n)$

$(v_1, 0, 0, \dots, 0)$

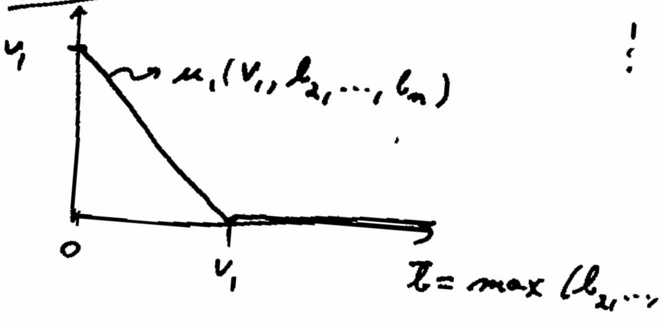
$(v_2, v_1, 0, \dots, 0)$

$(0, v_1, 0, \dots, 0)$

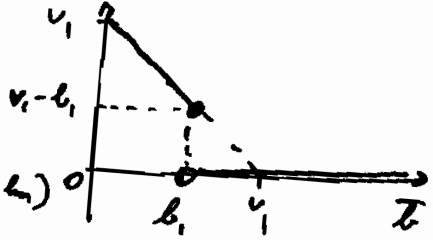
$(0, 0, \dots, 0, v_1)$

$\vdots$

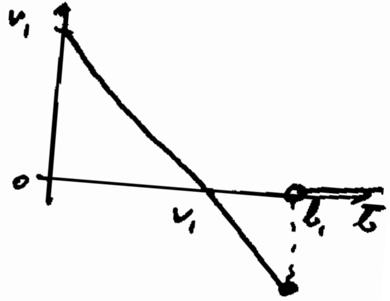
if $b_1 = v_1$:



If $b_1 < v_1$:



If $b_1 > v_1$:



$\Rightarrow$ $b_1 = v_1$ weakly dominates all the other bids.

"Distinguished" N.E. : $(b_1^*, \dots, b_n^*) = (v_1, \dots, v_n)$

Exercise 84.1

Exercise 85.1

Exercise 86.1

Set of N.E. $= \{(b_1, b_2) \mid b_1 \leq v_2 \text{ and } b_2 \geq v_1\}$
$\cup \{(b_1, b_2) \mid b_1 \geq v_2,\ b_1 \geq b_2, \text{ and } b_2 \leq v_1\}$.

3.5.3 First-price sealed-bid auction

$\left[\begin{array}{l}\text{Bidder } i\text{'s valuation} = v_i \\ v_1 > v_2 > \cdots > v_n > 0\end{array}\right.$

Players: $N = \{1, \ldots, n\}$

Actions: $A_1 = \cdots = A_n = [0, +\infty)$

Preferences: $u_i(b_1, \ldots, b_n) = v_i - b_i$, if $\left[\begin{array}{l} b_i = \max(b_1, \ldots, b_n) \\ \quad \text{and} \\ i \leq k, \text{ for all } k \text{ s.t. } b_k = \max(b_1, \ldots, b_n)\end{array}\right.$

$= 0$, otherwise.

Examples of N.E.: $(v_1, v_1, v_3, \ldots, v_n)$

$(v_2, v_2, 0, \ldots, 0)$

$(w, w, v_n, \ldots, v_n) \quad v_2 < w < v_1$

$\vdots$

$\boxed{\text{Exercise } 86.2}$

$\boxed{\text{Exercise } 87.1}$

Set of N.E.

$= \{(b, b, b_3, \ldots, b_n) \mid v_2 \leq b \leq v_1 \text{ and } b_k \leq b, \text{ for all } k \geq 3\}$

For bidder i:

$\left(\begin{array}{l} \text{any bid } b_i' > v_i \text{ is weakly dominated by } b_i = v_i \\ \text{the bid } b_i' = v_i \text{ is weakly dominated by } b_i < v_i \end{array}\right.$

in the case of a small monetary unit $\varepsilon > 0$:

$(v_2 - \varepsilon, v_2 - \varepsilon, b_3, \ldots, b_n)$ where $b_k \leq v_k - \varepsilon$, for all $k \geq 3$, are the only N.E. with bids that are not weakly dominated

if $\varepsilon \to 0$

"Distinguished" Nash Equilibria: $(v_2, v_2, b_3, \ldots, b_n)$, $b_k \leq v_k$ for all $k \geq 3$

- The distinguished N.E. of the second-price and the first-price auctions yield the same outcome.

3.5.4 Variants

Uncertain valuations

Common valuations

All-pay auctions $\boxed{\text{Exercise 89.1}}$

Multi-unit auctions:

Discriminatory auction

Uniform-price auction

Vickrey auction

$\boxed{\text{Exercise 90.1}}$ G

($\boxed{\text{Exercise 90.2}}$ G)

$\boxed{\text{Exercise 90.3}}$

Lobbying as an auction

$\boxed{\text{Exercise 91.1}}$

Kap. 4: Mixed Strategy Equilibrium

4.1 Introduction

4.1.1 Stochastic steady states

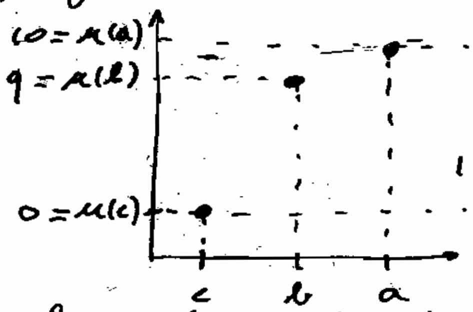
4.1.2 Example

		$\frac{1}{2}$ H	$\frac{1}{2}$ T
$\frac{1}{2}$	H	1,-1	-1,1
$\frac{1}{2}$	T	-1,1	1,-1

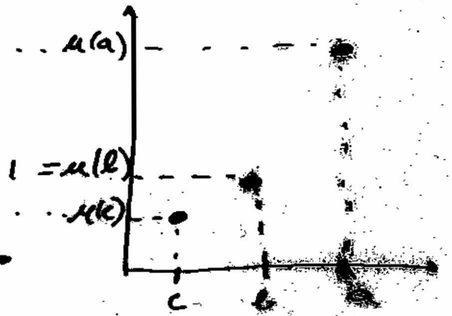
Exercise 101.1

4.1.3 Expected payoffs

Von Neumann – Morgenstern preferences
Bernoulli payoff function



Bernoulli payoff function for a person very averse to risk



Bernoulli payoff funct for a person not averse to risk

4.2 Strategic games in which players may randomize

Definition 106.1 Strategic game with vNM preferences

- Set of Players P
- for each player, a set of actions A_i , $i \in P$
- for each player, vNM preferences over lotteries of action profiles Bernoulli payoff function u_i , $i \in P$

Example:

	Q	F
Q	2,2	0,3
F	3,0	1,1

	Q	F
Q	3,3	0,4
F	4,0	1,1

As games with ordinal preferences: =

As games with vNM preferences: ≠

Exercise 106.2

4.3 Mixed strategy Nash equilibrium

4.3.1 Mixed strategies

Definition 107.1: Mixed strategy

A mixed strategy α_i of player i is a probability distribution over A_i. $\alpha_i \in \Delta(A_i)$

A mixed strategy α_i is pure iff there exists $a_i \in A_i$ s.t. $\alpha_i(a_i) = 1$
$\alpha_i(a_i') = 0$, for all $a_i' \in A_i$ s.t. $a_i' \neq a_i$

4.3.2 Equilibrium

Definition 108.1: Mixed strategy N.E. of strategic game with vNM preferenc

Let $G = (P, (A_i), (u_i))$ be a strategic game with vNM preferences
Let α^* be a mixed strategy profile of G $\left(\alpha^* \in \prod_{i \in P} \Delta(A_i)\right)$

α^* is a (mixed strategy) N.E. iff $\underbrace{U_i(\alpha^*)}_{\text{player } i\text{'s expected payoff}} \geq U_i(\alpha_i, \alpha^*_{-i})$, for all $\alpha_i \in \Delta(A_i)$ and all $i \in P$.

4.3.3 Best response functions

$$B_i(\alpha_{-i}) = \{\alpha_i \in \Delta(A_i) \mid U_i(\alpha_i, \alpha_{-i}) \geq U_i(\alpha_i', \alpha_{-i}), \text{ for all } \alpha_i' \text{ in } \Delta(A_i)\}$$

Proposition: α^* is NE iff $\alpha_i^* \in B_i(\alpha^*_{-i})$, for all $i \in P$.

Two-player two-action games: $\alpha_2 = (q, 1-q)$

$q = \alpha_2(L) \swarrow \quad \searrow \alpha_2(R) = 1-q$

$$(p, 1-p) = \alpha_1 \quad \begin{array}{l} p = \alpha_1(T) \nearrow \\ 1-p = \alpha_1(B) \searrow \end{array} \quad \begin{array}{c|c|c|} & L & R \\ \hline T & pq & p(1-q) \\ \hline B & (1-p)q & (1-p)(1-q) \\ \hline \end{array}$$

Bernoulli payoffs

	L	R
T	$u_1(T,L), u_2(T,L)$	$u_1(T,R), u_2(T,R)$
B	$u_1(B,L), u_2(B,L)$	$u_1(B,R), u_2(B,R)$

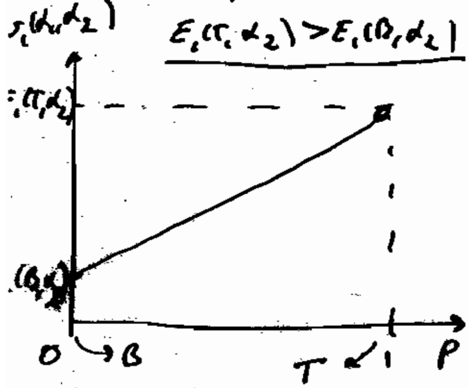
Expected payoffs:

$$v_1(\alpha_1,\alpha_2) = pq\,u_1(T,L) + p(1-q)\,u_1(T,R) + (1-p)q\,u_1(B,L) + (1-p)(1-q)\,u_1(B,R)$$

$$= p\underbrace{\left[q\,u_1(T,L) + (1-q)\,u_1(T,R)\right]}_{\parallel} + (1-p)\underbrace{\left[q\,u_1(B,L) + (1-q)\,u_1(B,R)\right]}_{\parallel}$$

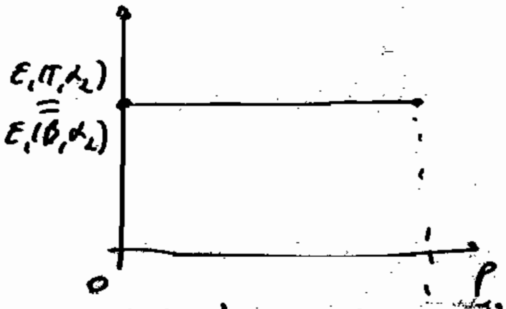
$$= p\,\overbrace{E_1(T,\alpha_2)} + (1-p)\,\overbrace{E_1(B,\alpha_2)}$$

$E_1(T,\alpha_2) > E_1(B,\alpha_2)$



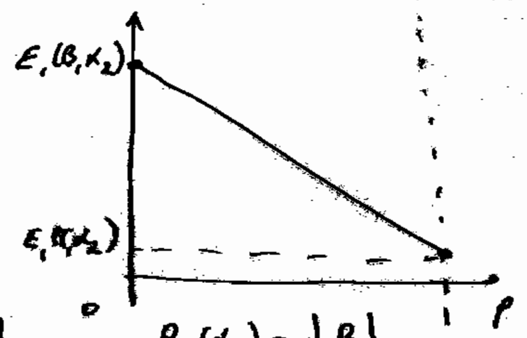
$B_1(\alpha_2) = \{T\}$

$E_1(T,\alpha_2) = E_1(B,\alpha_2)$



$B_1(\alpha_2) \neq \{(p,1-p) \mid 0 \le p \le 1\}$

$E_1(T,\alpha_2) < E_1(B,\alpha_2)$



$B_1(\alpha_2) = \{B\}$

Exercise 110.1

Exercise 111.1

1) Matching Pennies

H	1, -1	-1, 1
T	-1, 1	1, -1

$q \swarrow \ \searrow 1-q$

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

$E_1(H,q) = q \cdot 1 + (1-q)\cdot(-1) = 2q-1$

$E_1(T,q) = q\cdot(-1) + (1-q)\cdot 1 = 1-2q$

$\Rightarrow B_1(q) = \{0\}$ if $q < \frac{1}{2}$

$= [0,1]$ if $q = \frac{1}{2}$

$= \{1\}$ if $q > \frac{1}{2}$

	H	T
$p \nearrow$ H	1, -1	-1, 1
$1-p \searrow$ T	-1, 1	1, -1

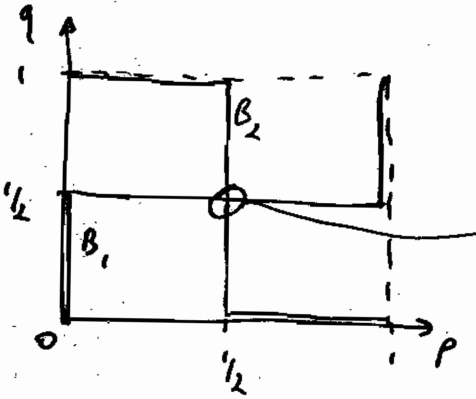
$E_2(H,p) = p(-1) + (1-p)\cdot 1 = 1-2p$

$E_2(T,p) = p\cdot 1 + (1-p)(-1) = 2p-1$

$\Rightarrow B_2(p) = \{1\}$ if $p < \frac{1}{2}$

$= [0,1]$ if $p = \frac{1}{2}$

$= \{0\}$ if $p > \frac{1}{2}$



N.E.: $(p^*, q^*) = (\frac{1}{2}, \frac{1}{2})$

2) BoS $q \swarrow \ \searrow 1-q$

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

$E_1(B,q) = q\cdot 2 + (1-q)\cdot 0 = 2q$

$E_1(S,q) = q\cdot 0 + (1-q)\cdot 1 = 1-q$

$\Rightarrow B_1(q) = \{0\}$ if $q < \frac{1}{3}$

$= [0,1]$ if $q = \frac{1}{3}$

$= \{1\}$ if $q > \frac{1}{3}$

	B	S
$p \nearrow$ B	2, 1	0, 0
$1-p \searrow$ S	0, 0	1, 2

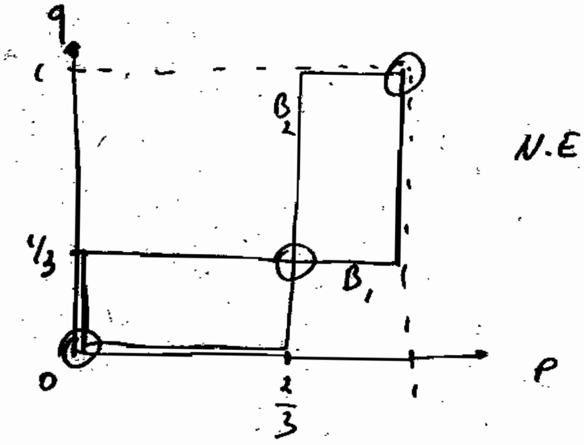
$E_2(B,p) = p\cdot 1 + (1-p)\cdot 0 = p$

$E_2(S,p) = p\cdot 0 + (1-p)\cdot 2 = 2-2p$

$\Rightarrow B_2(p) = \{0\}$ if $p < \frac{2}{3}$

$= [0,1]$ if $p = \frac{2}{3}$

$= \{1\}$ if $p > \frac{2}{3}$



N.E.: $(0,0)$, $(\frac{2}{3}, \frac{1}{3})$, $(1,1)$

$\downarrow$ (S,S) $\downarrow$ (B,B)

Exercise 114.1

Exercise 114.2

Exercise 114.3

Exercise 114.4

$$v_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i)\, E_i(a_i, \alpha_{-i})$$

$\Downarrow$

<u>Proposition</u>: characterization of mixed strategy NE of finite games

α^* is a N.E. iff. for all $i \in P$:

$$E_i(a_i, \alpha^*_{-i}) = E_i(a_i', \alpha^*_{-i}) \geq E_i(a_i'', \alpha^*_{-i}),\ \text{for all } a_i, a_i', a_i'' \in A_i$$

s.t.

$\alpha_i(a_i) > 0,\ \alpha_i(a_i') > 0$

$\alpha_i(a_i'') = 0$.

equilibrium conditions

if α^* is a N.E., then $v_i(\alpha^*) = E_i(a_i, \alpha^*_{-i})$, for all $i \in P$ and all $a_i \in A_i$ s.t. $\alpha_i^*(a_i) > 0$.

<u>Example</u>:

$$\begin{array}{c|c|c|c|}
 & L(0) & C(\frac{1}{3}) & R(\frac{2}{3}) \\
\hline
(\frac{3}{4})\ T & 0,2 & 3,3 & 1,1 \\
\hline
(0)\ M & 0,0 & 0,0 & 2,0 \\
\hline
(\frac{1}{4})\ B & 0,4 & 5,1 & 0,7 \\
\hline
\end{array}$$

$$\alpha^* = \left((\tfrac{3}{4}, 0, \tfrac{1}{4}), (0, \tfrac{1}{3}, \tfrac{2}{3}) \right)$$

$$\left[\begin{array}{l}
E_1(T, \alpha_2^*) = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3} \\
E_1(M, \alpha_2^*) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3} \\
E_1(B, \alpha_2^*) = \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}
\end{array}\right. \Rightarrow \begin{array}{l} E_1(T, \alpha_2^*) = E_1(B, \alpha_2^*) \\ \geq E_1(M, \alpha_2^*) \end{array}$$

$$\left[\begin{array}{l}
E_2(L, \alpha_1^*) = \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{5}{2} \\
E_2(C, \alpha_1^*) = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{5}{2} \\
E_3(R, \alpha_1^*) = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 7 = \frac{5}{2}
\end{array}\right. \Rightarrow \begin{array}{l} E_2(C, \alpha_1^*) = E_2(R, \alpha_1^*) \\ \geq E_2(L, \alpha_1^*) \end{array}$$

$\Downarrow$

α^* is a N.E.

and

$v_1(\alpha^*) = \frac{5}{3}$, $v_2(\alpha^*) = \frac{5}{2}$

<u>Exercise 117.2</u>

<u>Exercise 118.1</u>

(<u>Exercise 118.2</u> G)

<u>Exercise 48.3</u>

4.3.5 Existence of equilibrium in games

Proposition Existence of mixed strategy NE in finite games

- Let $G=(P,(A_i),(u_i))$ be a strategic game with vNM preferences s.t. $\# A_i < +\infty$, for all $i \in P$.

Then, there exists a (mixed strategy) NE α^* of G.

4.4 Dominated Actions (Strategies)

Definition 120.1 : Strict domination

Let $G=(P,(A_i),(u_i))$ be a strategic game with vNM preferences.

Let i, a_i', α_i (α_i') be s.t. $i \in P$, $a_i' \in A_i$ $(\alpha_i' \in \Delta(A_i))$, $\alpha_i \in \Delta(A_i)$.

α_i strictly dominates a_i' iff $U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i})$ $(U_i(\alpha_i', a_{-i}))$, for all $a_{-i} \in A_{-i}$

Example: $u_1(a_1, a_2)$:

	L	R
T	1	1
M	4	0
B	0	3

$(0, \frac{1}{2}, \frac{1}{2})$ st. dominates T.

Exercise 120.2

Exercise 120.3

Proposition :

A strictly dominated action is not used with positive probability in any mixed strategy N.E.

Definition 121.1 : Weak domination

α_i weakly dominates a_i' (α_i') iff $U_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i})$ $(U_i(\alpha_i', a_{-i}))$, for all $a_{-i} \in A_{-i}$

and

there exists $a_{-i}' \in A_{-i}$ s.t.

$U_i(\alpha_i, a_{-i}') > u_i(a_i', a_{-i}')$. $(U_i(\alpha_i', a_{-i}'))$

Exercise 121.2

Proposition: Existence of mixed strategy NE with no weakly dominated strategies in finite games

Let $G=(P,(A_i),(u_i))$ be a strategic game with vNM preferences s.t. $\# A_i < +\infty$, for $i \in P$.

Then, there exists a NE α^* of G s.t. α_i^* is not weakly dominated, for all $i \in P$.

4.5 Pure equilibria when randomization is allowed

Bernouilli payoff function

Let $G' = (P, (A_i), (u_i))$ be a strategic game with vNM preferences

Let $G = (P, (A_i), (u_i))$ be the corresponding strategic game with ordinal preferences

Proposition 122.2 Pure strategy equilibria survive when randomization is allowed

Let a^* be a NE of G.

Let α^* be the (mixed) strategy profile of G' s.t. , for all $i \in P$,

$$\alpha_i^*(a_i^*) = 1, \ \alpha_i^*(a_i') = 0 \text{ for all } a_i' \in A_i, \ a_i' \neq a_i^*.$$

Then α^* is a mixed strategy NE of G'.

Proposition 123.1 Pure strategy equilibria survive when randomization is prohibited.

Let α^* be a (mixed) strategy NE of G' and let a^* be a strategy profile of G st for all $i \in P$: $\alpha_i^*(a_i^*) = 1$, $\alpha_i^*(a_i') = 0$ for all $a_i' \in A_i$ st. $a_i' \neq a_i^*$.

Then a^* is a NE of G. (4.6 Expert diagnosis)

4.7 Equilibrium in a single population

Definition 129.1 : Symetric 2-player strategic game with vNM preferences.

Let $G = (P = \{1, 2\}, (A_i), (u_i))$ be a strategic game with vNM preferences between 2 players.

G is symetric iff $\left[\begin{array}{l} A_1 = A_2 \\ u_1(a_1, a_2) = u_2(a_2, a_1), \text{ for all } a_1, a_2 \in A_1 = A \end{array}\right.$

Definition 129.2 : Symetric mixed strategy NE

Let $G = (P, (A_i), (u_i))$ be a strategic game with vNM preferences s.t. $A_1 = \dots = A_n$

Let α^* be a mixed strategy profile of G.

α^* is a symmetric mixed strategy NE of G iff. $\left[\begin{array}{l} \alpha^* \text{ is a mixed strategy NE of } G \\ \alpha_1^* = \dots = \alpha_n^* \end{array}\right.$

Examples:

	L	R
L	1,1	0,0
R	0,0	1,1

Symmetric NE:

$((1,0),(1,0)) = (L,L)$ } pure strategy NE
$((0,1),(0,1)) = (R,R)$

$((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$

	X	Y
X	0,0	1,1
Y	1,1	0,0

Symmetric NE:

$((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$

Proposition 130.1 Existence of symmetric NE in symmetric finite games

Let $G=(P=\{1,2\}, (A_i), (a_i))$ be a symmetric 2-player strategic game with vNM preferences s.t. $\# A_1 = \# A_2 < +\infty$.
Then, there exists a symmetric mixed strategy profile α^* of G.

Exercise 130.2

Exercise 130.3

4.8 Illustration

Players : $P=\{1,\dots,n\}$

Actions : $A_1 = \dots = A_n = \{C, D\}$

Preferences : $u_i(D,\dots,D)=0$, $u_i(C,a_{-i}) = v-c$, $u_i(D,a_{-i}) = v$
$(D,a_{-i}) \neq (D,\dots,D)$

$\swarrow$ Bernoulli payoff function

for all $i=1,\dots,n$

with $v>c>0$.

n pure N.E.: $(C,D,\dots,D)$, $(D,C,D,\dots,D)$, ..., $(D,\dots,D,C$

not symmetric

<u>Symmetric mixed strategy NE</u> : $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$ with $\alpha_1^* = \dots = \alpha_n^*$.

Let $\alpha_1^*(C) = \dots = \alpha_n^*(C) = p$, with $0 < p < 1$.

Equilibrium condition: $\underbrace{E_i(C, \alpha_{-i}^*)} = E_i(D, \alpha_{-i}^*)$

$$v - c = 0 \cdot \text{Prob}\{\text{no player } j \neq i \text{ calls}\} + v \cdot \text{Pr}\{\text{at least one player } j \neq i \text{ calls}\}$$

$$\Downarrow$$

$$v - c = v\,(1 - \text{Prob}\{\text{no player } j \neq i \text{ calls}\})$$

$$\Downarrow$$

$$\frac{c}{v} = \text{Pb}\{\text{no player } j \neq i \text{ calls}\}$$

$$\Rightarrow \quad \frac{c}{v} = (1-p)^{n-1} \qquad \Rightarrow \qquad p = 1 - \left(\frac{c}{v}\right)^{1/n-1}$$

[Symmetric N.E. α^*
with $\alpha_i^* = \left(1 - \left(\frac{c}{v}\right)^{1/n-1}, \left(\frac{c}{v}\right)^{1/n-1}\right)$
for all $i = 1, \dots, n$

<u>Remarks</u>:

- $\alpha_i^*(C) = p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$ ↓ if n ↑
- Prob{no one calls} = Prob{i does not call} Prob{no player j ≠ i calls}

$$= \left(\frac{c}{v}\right)^{\frac{1}{n-1}} \cdot \frac{c}{v} \quad \uparrow \text{ if } n \uparrow.$$

<u>Exercise 132.2</u>

<u>Exercise 132.3</u>

(4.9 Formation of players' beliefs)

4.10 Finding all mixed strategy NE

For each $(S_1, \dots, S_n)$ s.t. $S_i \subseteq A_i$ and $S_i \neq \phi$, for all i,

look for α^* s.t. $\Bigg($ $\alpha_i^*(a_i) > 0$ and $\alpha_i^*(a_i') = 0$, for all $a_i \in S_i$, $a_i' \notin S_i$

α^* satisfies equilibrium conditions

Example 138.1

Exercise 139.1

Example 139.2

	B	S	X
B	4, 2	0, 0	0, 1
S	0, 0	2, 4	1, 3

$\#S_1 = \#S_2 = 1$: Look for pure strategy NE.

(B, B) (S, S)

$\#S_1 = 1$, $\#S_2 = 2$:

$S_1 = \{B\}$: $u_2(B,B) = 2 \neq u_2(B,S) = 0 \neq u_2(B,X) = 1$

⇒ "1st equilibrium condition" cannot be satisfied.
⇒ no such NE

$S_1 = \{S\}$: $u_2(S,B) = 0 \neq u_2(S,S) = 4 \neq u_2(S,X) = 3$

⇒ "1st equilibrium condition" cannot be satisfied.
⇒ no such NE

$\#S_1 = 2$, $\#S_2 = 1$:

$S_2 = \{B\}$: $u_1(B,B) = 4 \neq u_1(S,B) = 0$ → no such NE

$S_2 = \{S\}$: $u_1(B,S) = 0 \neq u_1(S,S) = 2$ → no such NE

$S_2 = \{X\}$: $u_1(B,X) = 0 \neq u_1(S,X) = 1$ → no such NE

$\#S_1 = 2$, $\#S_2 = 2$:

$\Downarrow$

$S_1 = \{B, S\}$
$\alpha_1^*(B) = p \in (0,1)$

$S_2 = \{B, S\}$: $\alpha_2^*(B) = q \in (0,1)$

1st equ. cond. : $E_2(B, \alpha_1^*) = E_2(S, \alpha_1^*) \rightarrow 2p = 4(1-p) \longrightarrow p = \frac{2}{3}$

2nd ——— : $E_2(S, \alpha_1^*) \geq E_2(X, \alpha_1^*) \rightarrow 4(1-p) \geq p + 3(1-p)$ $\quad 4 \cdot \frac{1}{3} < \frac{2}{3} +$

for player 2

no such NE

$\underline{S_2 = \{B, X\}}$: $\alpha_2^*(B) = q \in (0,1)$

$$\left(\begin{array}{l} E_2(B,\alpha_1^*) = E_2(X,\alpha_1^*) \to 2p = p+3(1-p) \longrightarrow p = \frac{3}{4} \\ E_2(X,\alpha_1^*) \geq E_2(S,\alpha_1^*) \to p+3(1-p) \geq 4(1-p) \quad \frac{6}{4} \geq 4 \cdot \frac{1}{4} \end{array} \right. \to \alpha_1^* = \left(\frac{3}{4}, \frac{1}{4}\right)$$

1^{st} eq. cond : $E_1(B,\alpha_2^*) = E_1(S,\alpha_2^*) \to 4q = 1-q \to q = \frac{1}{5}$

2^{d} ——— : ✓

for player 1

$\to \alpha_2^* = \left(\frac{1}{5}, 0, \frac{4}{5}\right)$

N.E. : $\left(\left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{1}{5}, 0, \frac{4}{5}\right)\right)$

$\underline{S_2 = \{S, X\}}$: $\alpha_2^*(S) = q \in (0,1)$

$u_1(B,S) = 0 < u_1(S,S) = 2$

$u_1(B,X) = 0 < u_1(S,X) = 1$ $\Rightarrow E_1(B,\alpha_2^*) < E_1(S,\alpha_2^*)$

↓

1^{st} eq. cond. for player 1 is not satisfied

↪ no such NE

$\underline{\# S_1 = 2, \ \# S_2 = 3}$: $S_1 = \{B, S\}$, $S_2 = \{B, S, X\}$

$\alpha_1^*(B) = p \in (0,1)$

1^{st} eq. cond for player 2 : $E_2(B,\alpha_1^*) = E_2(S,\alpha_1^*) = E_2(X,\alpha_1^*)$

2^{d} ——— : ✓

$$\begin{array}{ccccc} 2p & = & 4(1-p) & = & p+3(1-p) \\ \downarrow & & & & \downarrow \\ p = \frac{2}{3} & & \neq & & \frac{1}{2} = p \end{array}$$

↪ no such NE

$\Rightarrow$ $\underline{3 \text{ NE}}$:

$\left((1,0), (1,0,0)\right) \longrightarrow (B, B)$

$\left((0,1), (0,1,0)\right) \longrightarrow (S, S)$

$\left(\left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{1}{5}, 0, \frac{4}{5}\right)\right)$

<u>Exercise 141.1</u>

<u>Exercise 141.2</u>

(<u>Exercise 141.3</u> G)

<u>Exercise 142.1</u>

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LEBRUN

Chapter 5: Extensive games with perfect information: Theory

5.1 Extensive games with perfect information

Definition 155.1 Extensive game with perfect information

- Set of players N
- Set of terminal histories Z
- A player function $P : \{\text{proper subhistories}\} \longrightarrow N$
  $h \longrightarrow P(h) \in N$
- for each player, preferences over the set of terminal histories $\succsim_i$ over Z
  $\hookrightarrow$ or u_i

Example (Entry game)

- $N = \{1, 2\}$
  1 → challenger
  2 → incumbent
- $Z = \{(I, A), (I, F), Out\}$
  Out → out
  A → Acquiesce
  I → In
  F → Fight

$H = \{\text{histories}\}$

$\quad = \{\phi, I, O, (I, A), (I, F)\}$

$H \setminus Z = \{\text{proper subhistories}\}$

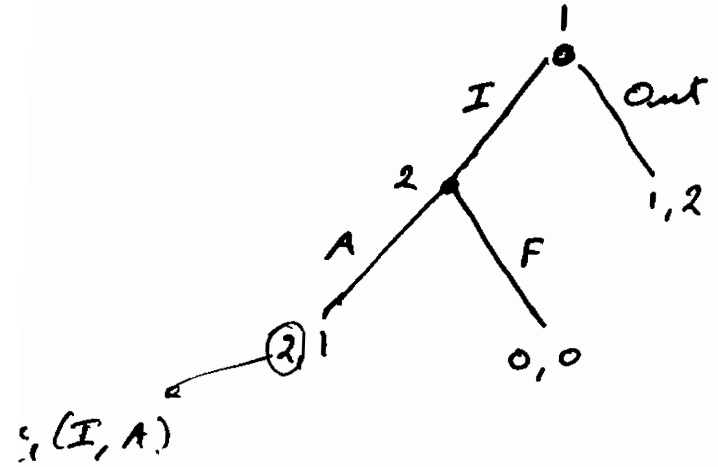
$\quad = \{\phi, I\}$

- $P : H \setminus Z \longrightarrow N$
  $\phi \longrightarrow P(\phi) = 1$
  $I \longrightarrow P(I) = 2$
- $u_1(I, A) = 2$, $u_1(Out) = 1$, $u_1(I, F) = 0$
  $u_2(Out) = 2$, $u_2(I, A) = 1$, $u_2(I, F) = 2$

For all $h \in H \setminus Z$, $A(h) = \{a \mid (h, a) \in H\}$

$A(h)$ → Set of actions available to player moving at h

Here: $A(\phi) = \{I, Out\}$, $A(I) = \{A, F\}$



Exercise 156.2

5.2 Strategies and outcomes

5.2.1 Strategies

Definition 159.1 Strategy

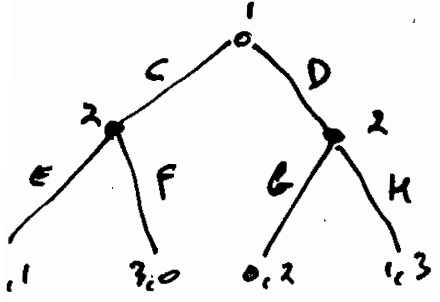
Let (N, H, P, u_i) be an extensive game with perfect information

s_i is a strategy of player i iff $s_i : \underbrace{\{h \in H\setminus Z \mid P(h)=i\}}_{\text{set of histories after}} \longrightarrow \bigcup_{h \in H\setminus Z} A(h)$

player i moves

$h \longrightarrow s_i(h) \in A(h)$

Example:



$H\setminus Z = \{\phi, C, D\}$ $P(\phi)=1$, $P(C)=P(D)=2$

$A(\phi)=\{C,D\}$

$A(C)=\{E,F\}$

$A(D)=\{G,H\}$

Strategies of Player 1:

$s_1 : \{\phi\} \longrightarrow \{C, D\}$

$\phi \longrightarrow C$ C

$s_1' : \{\phi\} \longrightarrow \{C, D\}$

$\phi \longrightarrow D$ D

Strategies of Player 2:

${}_2 : \{C, D\} \longrightarrow \{E, F, G, H\}$ EG

$C \longrightarrow s_2(C) = E \in \{E, F\}$

$D \longrightarrow s_2(C) = G \in \{G, H\}$

$s_2 : C \longrightarrow E$ EH

$D \longrightarrow H$

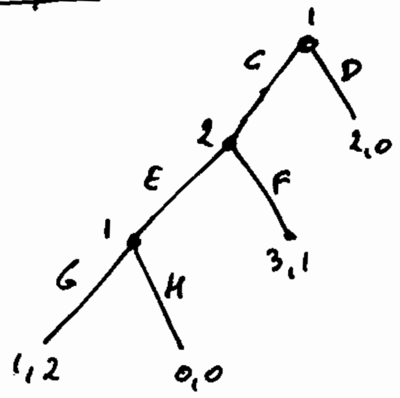
${}_2 : C \longrightarrow F$ FG

$D \longrightarrow G$

$s_2 : C \longrightarrow F$ FH

$D \longrightarrow H$

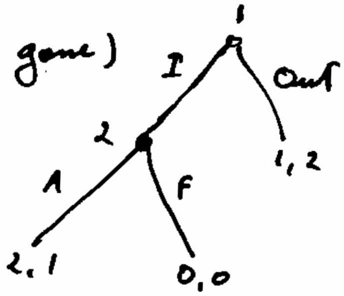
Examples:



Player 1's strategies: CG, CH, DG, DH

Player 2's strategies: E, F

(Entry game)



Player 1's strategies: I, Out

Player 2's strategies: A, F

Exercise (61.1)

5.2.2 Outcomes

$s = (s_1, \dots, s_n)$ strategy profile

↳ Outcome of s = $O(s) = (a^1, a^2, a^3, \dots) \in Z$ → terminal history if players follow $(s_1, \dots, s_n)$

$s_{P(\emptyset)}(\emptyset)$ $\quad$ $s_{P(a^1)}(a^1)$ (if $a^1 \in H \setminus Z$) $\quad$ $s_{P(a^1, a^2)}(a^1, a^2)$ (if $(a^1, a^2) \in H \setminus Z$) $\dots$

Examples:

→ $O(DH, F) = O(DG, F) = O(DH, E) = O(DG, E) = D$

$O(CG, E) = (C, E, G)$, $O(CH, E) = (C, E, H)$

$O(CG, F) = O(CH, F) = (C, F)$

5.3 Nash Equilibrium

Definition 161.2 Nash equilibrium of extensive game with perfect information

If $\Gamma = (N, Z, P, \mu_i)$ be an extensive game with perfect information.

Let $s^* = (s_1^*, \dots, s_n^*)$ be a strategy profile of Γ

s^* a N.E. of Γ iff $\mu_i(O(s^*)) \geq \mu_i(O(r_i, s_{-i}^*))$, for all strategy r_i of player i, for all $i \in N$.

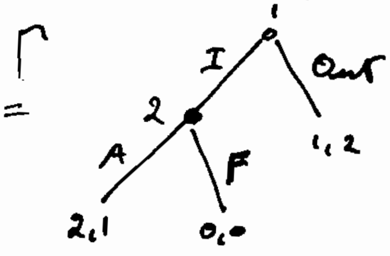
Strategic form of the extensive game Γ

$$\Gamma^{st.} = (N, \underbrace{A_i}_{S_i = \{\text{strategies of player } i \text{ in } G\}}, \mu_i^{st.}) \qquad \mu_i^{st.}(s_1, \dots, s_n) = \mu_i(O(s_1, \dots, s_n))$$

s^* is a N.E. of G iff s^* is a N.E. of $\Gamma^{st.}$

Exple: (entry game)

$\Gamma =$



$S_1 = \{\text{strategies of player } 1\} = \{I, Out\}$

$S_2 = \{\text{strategies of player } 2\} = \{A, F\}$

$\Gamma^{st.}$

Player 2

Player 1	A	F
I	2, 1	0, 0
Out	1, 2	1, 2

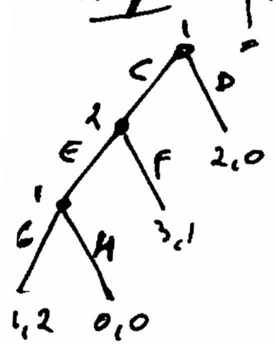
→ $\mu_1(IF), \mu_2(IF)$

$\mu_2(Out)$

N.E.: (I, A), (Out, F)

Example :

Γ =



Γ^{st} =

	Player 2 E	F
CG	1,2	3,1
CH	0,0	3,1
DG	2,0	2,0
DH	2,0	2,0

N.E. : (CH, F)
(DG, E)
(DH, E)

Exercise 163.1

5.4 Subgame perfect equilibrium

5.4.1 Definitions

Definition 164.1 Subgame of extensive game with perfect information

Let $\Gamma = (N, Z, P, u_i)$ be an extensive game with perfect information

If $h \in H \setminus Z$:

the subgame $\Gamma(h)$ is the extensive game with perfect information

$$(N, Z_h, P_h, u_{i,h})$$

with

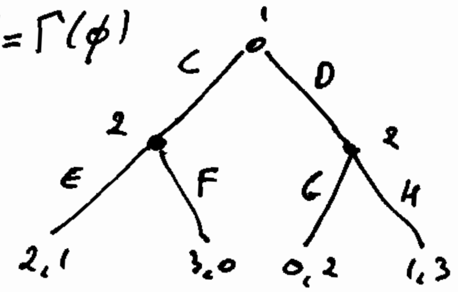
$$Z_h = \{ h' \mid (h, h') \in Z \}$$

$$P_h(h') = P(h, h') \text{ , for all } h' \text{ s.t. } (h, h') \in H \setminus Z$$

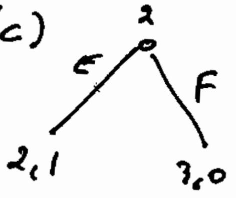
$$u_{i,h}(h') = u_i(h, h') \text{ , for all } h' \text{ s.t. } (h, h') \in Z$$

examples:

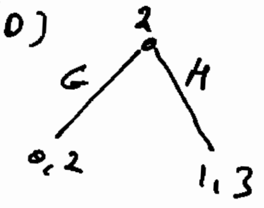
$\Gamma = \Gamma(\phi)$



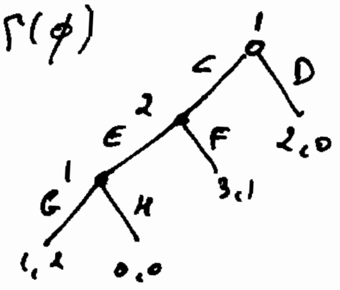
$\Gamma(C)$



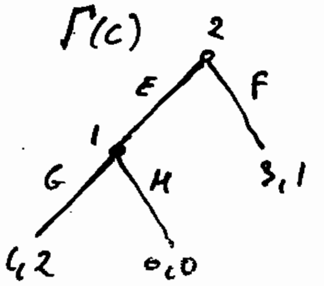
$\Gamma(D)$



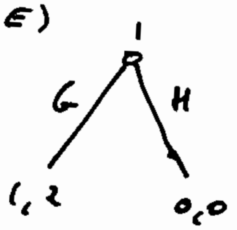
$= \Gamma(\phi)$



$\Gamma(C)$



$\Gamma(C, E)$



Exercise 164.2

Definition 166.1 Subgame perfect equilibrium of extensive game with perfect information

Let $\Gamma = (N, Z, P, u_i)$ be an extensive game with perfect information.

Let $s^* = (s_1^*, \dots, s_n^*)$ be a strategy profile of Γ.

s^* is a subgame perfect equilibrium of Γ iff $u_i(O_h(s^*)) \geq u_i(O_h(r_i, s_{-i}^*))$,

for all strategy r_i of player i,
for all $h \in H \setminus Z$ s.t. $P(h) = i$,
for all $i \in N$

where $O_h(s)$ is the outcome of s in $\Gamma(h)$.

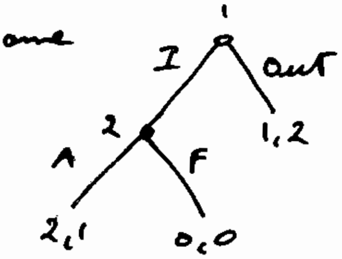
5.4.2 Subgame perfect equilibrium and Nash equilibrium

s^* is a subgame perfect equilibrium of Γ

iff

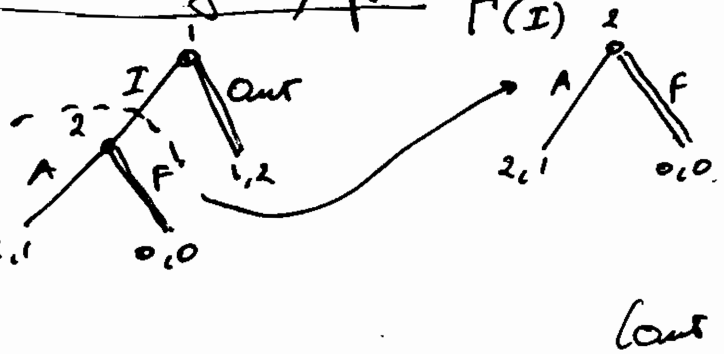
$s^*|_h$ is a N.E. of $\Gamma(h)$, for all $h \in H \setminus Z$

Example: entry game



2 N.E.: (I, A)
(out, F)

(out, F) not subgame perfect:



$h = I$, $P(I) = 2$

$u_2(O_I(F, out)) = 0 < 1 = u_2(O_I(A, out))$

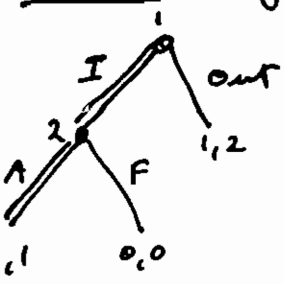
or

F is not a N.E. of $\Gamma(I)$

$\Downarrow$

(out, F) not subgame perfect

(I,A) is subgame perfect:



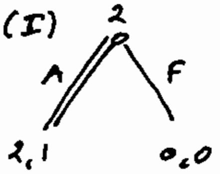
$h=\phi$, $P(\phi)=1$

$$u_1(O(I,A))=2 \geq u_1(O(\text{out},A))=1$$

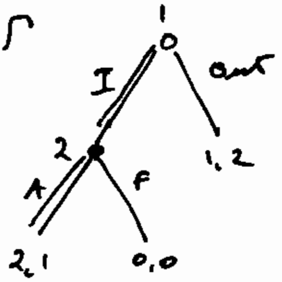
$h=I$, $P(h)=2$

$$u_2(O_I(I,A))=1 \geq u_2(O_I(I,F))=0$$

A is N.E. of $\Gamma(I)$



and (I,A) N.E. of $\Gamma(\phi)=\Gamma$



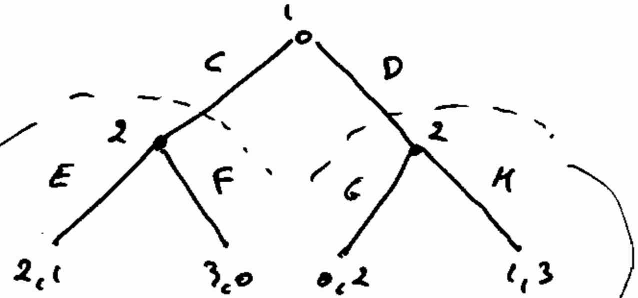
⇓

(I,A) is subgame perfect.

Exercise 168.1

5.5 Finding subgame perfect equilibria of finite horizon games: backward induction

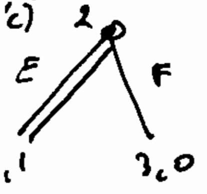
Examples: Γ

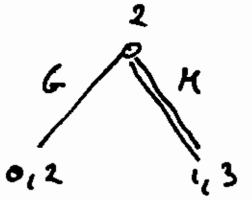


ep I

lgames of length 1:

(C)

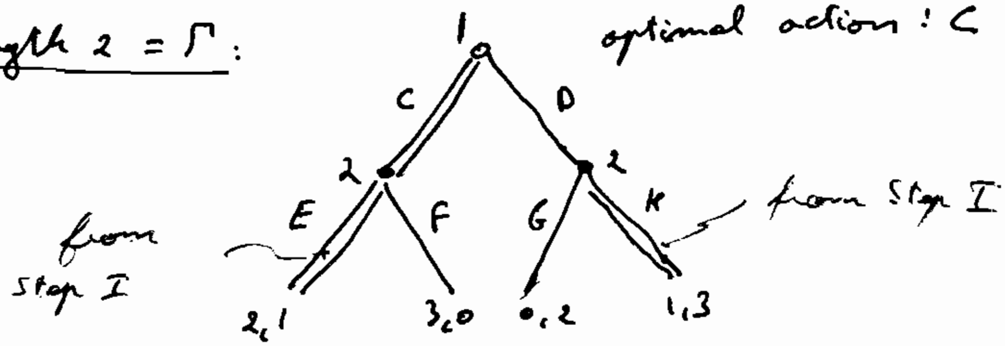




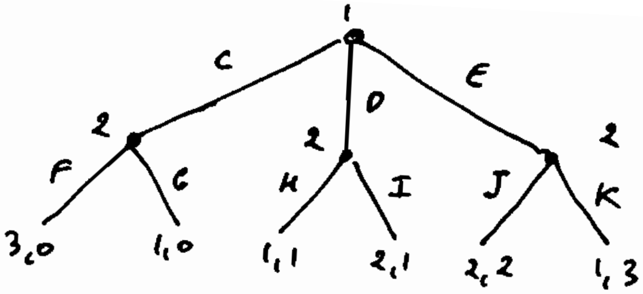
al
ions : E H

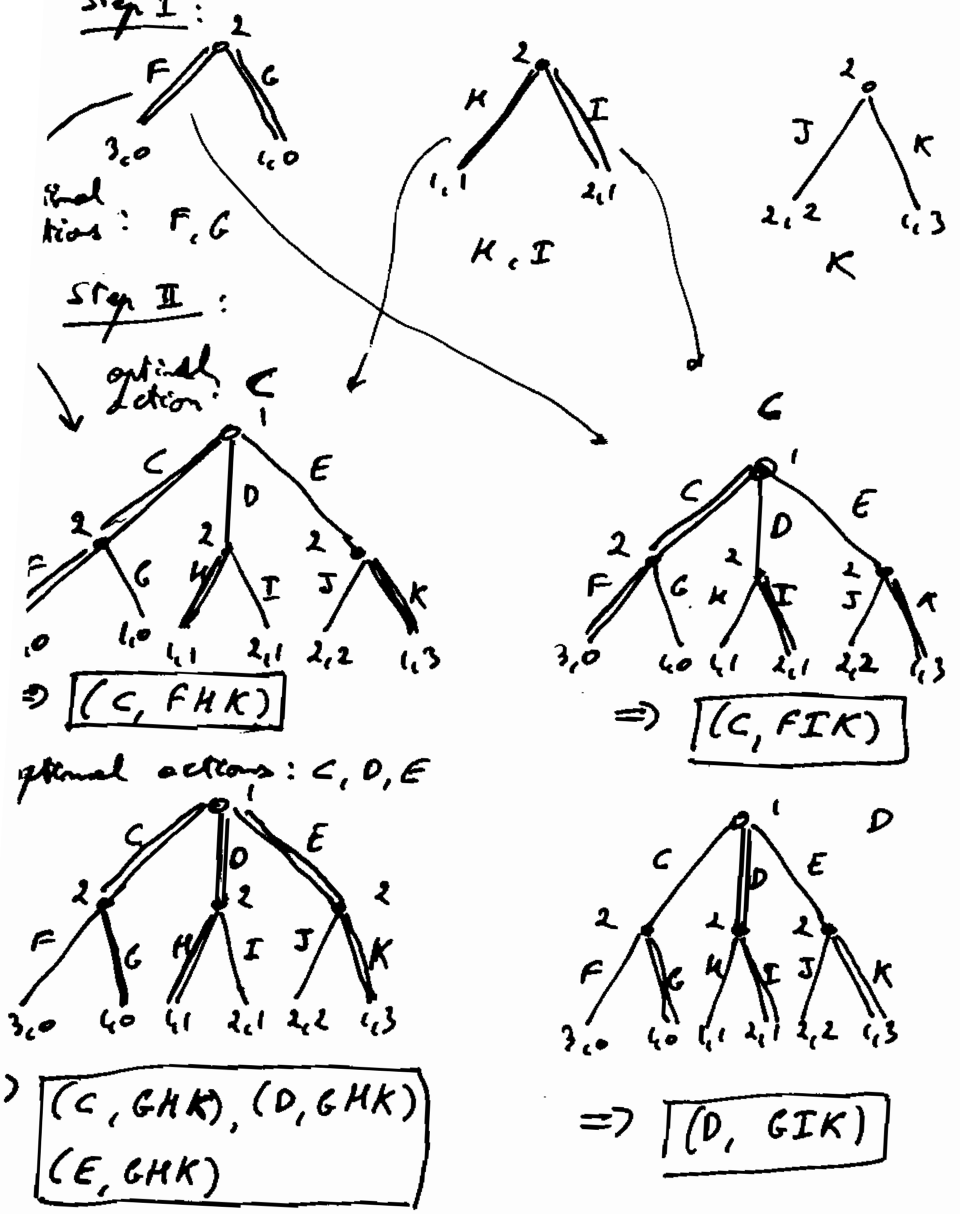
II Subgame of length 2 = Γ:

optimal action: C



→ (C, EH)





Proposition 172.1: Subgame perfect equilibrium of finite horizon games and backward induction

If Γ is a finite horizon extensive game with perfect information,

$\{$subgame perfect equilibria of $\Gamma\} = \{$results of the procedure of backward induction applied to $\Gamma\}$

oposition 173.1: Existence of subgame perfect equilibrium

If Γ is a finite extensive game with perfect information, then there

Exercise 173.2

(Exercise 173.3)

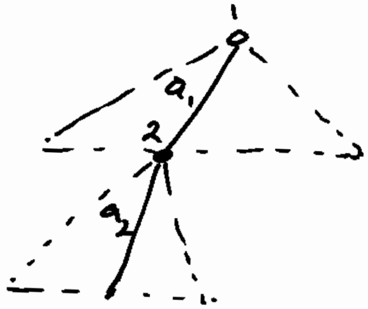
Exercise 173.4

Exercise 174.1

Exercise 174.2 (G)

Exercise 176.1

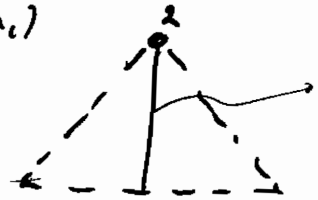
Example 176.2 A synergistic relationship



$a_1(c+a_2-a_1),\ a_2(c+a_1-a_2)$

Backward induction:

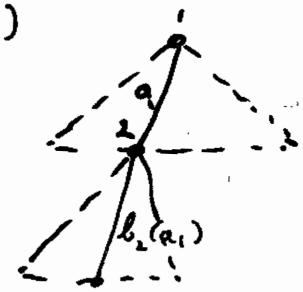
$\Gamma(a_1)$



$b_2(a_1) = \frac{1}{2}(c+a_1)$

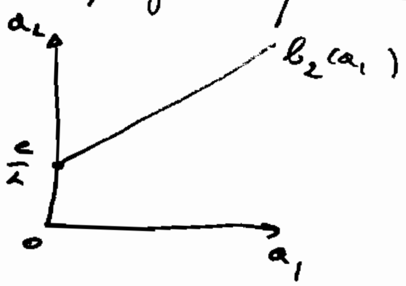
strategy of player 2 in any subgame perfect equilibrium

$\Gamma(\phi)$



$$\underbrace{a_1(c+b_2(a_1)-a_1)}_{\parallel},\ b_2(a_1)(c+a_1-b_2(a_1))$$

$$\frac{1}{2}a_1(3c-a_1)$$



In any subgame perfect equilibrium: $s_1^* = a_1^* \in \arg\max_{a_1} \frac{1}{2}a_1(3c-a_1)$

$\Rightarrow a_1^* = \frac{3}{2}c$

One Subgame perfect equilibrium: (s_1^*, s_2^*)

where $s_1^* = \frac{3}{2}c$

$s_2^* : \{a_1\} = [0,+\infty) \longrightarrow [0,+\infty) = \{a_2\}$

$a_1 \longrightarrow s_2^*(a_1) = \frac{1}{2}(c+a_1)$

Exercise 177.1 (exercise 177.2) Exercise 177.3

Chapter 6 Extensive Games with Perfect Information: Illustrations

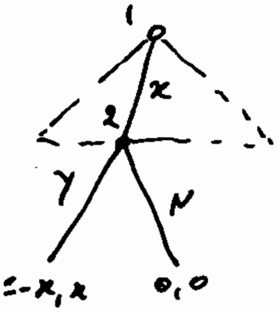
6.1.1 The ultimatum game : $c > 0$

$N = \{1,2\}$ Set of players

$Z = \{(x,Y),(x,N) \mid x \in [0,c]\}$ Set of terminal histories

$P(\phi) = 1, \quad P(x) = 2$ Player function

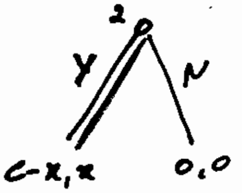
$$\begin{cases} u_1(x,Y) = c-x, & u_1(x,N) = 0 \\ u_2(x,Y) = x, & u_2(x,N) = 0 \end{cases}$$ Payoff functions



Backward induction:

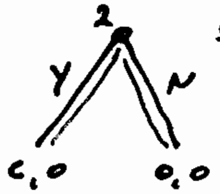
$\Gamma(x)$, with $x > 0$

$s_2^*(x) = Y$

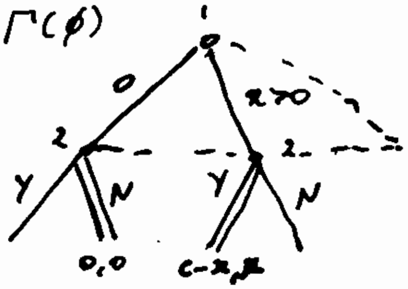


$\Gamma(x)$, with $x = 0$

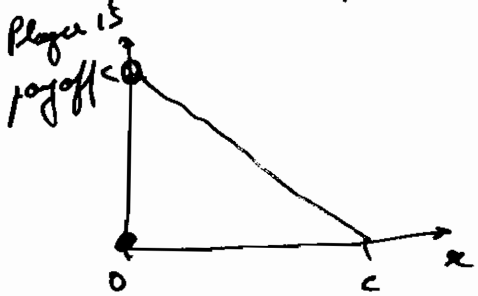
$s_2^*(x) = Y$ or N



Suppose $s_2^*(0) = N$: $\Gamma(\phi)$

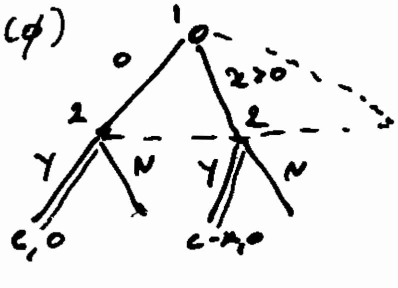


Player 1 has no best response



⇒ no such subgame perfect equilibrium

Suppose $s_2^*(0) = Y$: $\Gamma(\phi)$



Player 1 has a unique best response : $s_1^* = 0$

⇒ unique subgame perfect equilibrium: (s_1^*, s_2^*) with $s_1^* = 0$
s_2^* s.t. $s_2^*(x) = Y$, for all x

Exercise 183.1

Exercise 183.2

Exercise 183.3

(Exercise 183.4, 6)

Exercise 185.1

Exercise 185.2 (G)

6.1.2 The holdup game:

$c_H > c_L > 0$

$H > L > 0$

Exercise 186.1

$N = \{1, 2\}$

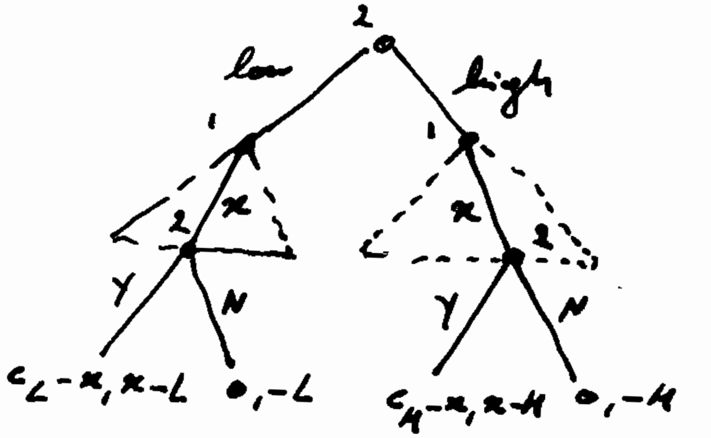
$Z = \{(low, x, Y), (low, x, N) \mid x \in [0, c_L]\} \cup \{(High, x, Y), (High, x, N) \mid x \in [0, c_H]\}$

$P(\phi) = 2$, $P(low) = P(High) = 1$, $P(low, x) = 2$, for all $x \in [0, c_L]$

$P(High, x) = 2$, for all $x \in [0, c_H]$

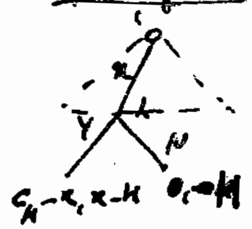
For all $x \in [0, c_L]$: $u_1(low, x, Y) = c_L - x$, $u_1(low, x, N) = 0$; $u_2(low, x, Y) = x - L$, $u_2(low, x, N) = -L$

For all $x \in [0, c_H]$: $u_1(high, x, Y) = c_H - x$, $u_1(high, x, N) = 0$; $u_2(high, x, Y) = x - H$, $u_2(high, x, N) = -H$



$\Gamma(low)$

$\Gamma(high)$:



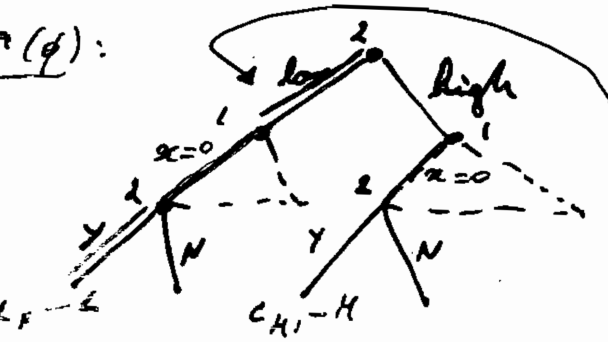
ultimatum game.

⇒ only subgame perfect equilibrium of $\Gamma(low)$, $\Gamma(high)$

s_1^*: $x = 0$

s_2^*: $s_2^*(x) = Y$, for all x

$\Gamma(\phi)$:



Player 2 has a unique best response:

$s_2^* = low$

⇒ unique subgame perfect equilibrium (s_1^*, s_2^*) with

$s_1^*(low) = s_1^*(high) = 0$

$s_2^*(\phi) = low$, $s_2^*(low, x) = Y$, for all $x \in [0, c_L]$

(6.1.3 Agenda control)

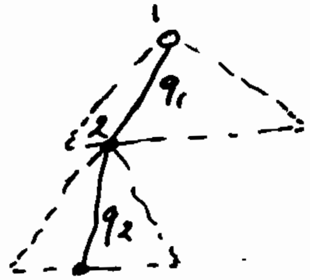
6.2 Stackelberg's model of oligopoly

6.2.1 General model :

Firm i's cost function C_i

Inverse demand function $P_d(Q)$

$$\left[\begin{array}{l} N = \{1, 2\} \\ Z = \{(q_1, q_2) \mid q_1, q_2 \geq 0\} \\ P(\phi) = 1 , \quad P(q_1) = 2 , \text{ for all } q_1 \geq 0 \\ u_i(q_1, q_2) = q_i P_d(q_1 + q_2) - C_i(q_i), \text{ for all } q_1, q_2 \geq 0 , \ i = 1, 2 \end{array}\right.$$



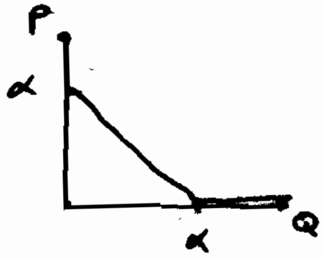
$q_1 P_d(q_1+q_2) - C_1(q_1), \ q_2 P_d(q_1+q_2) - C_2(q_2)$

6.2.2 Example :

$C_i(q_i) = c q_i$, for all $q_i \geq 0$, $i = 1, 2$

$$P_d(Q) = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases}$$

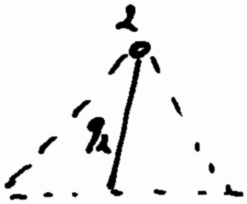
$0 < c < \alpha$



Γ

$, (\alpha - q_1 - q_2) - c q_1 , q_2 (\alpha - q_1 - q_2) - c q_2$

$\Gamma(q_1)$:



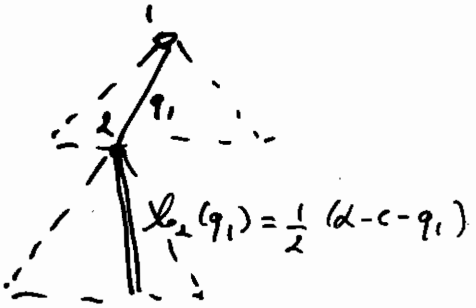
$q_2(\alpha - q_1 - q_2) - c q_2$

N.E. of $\Gamma(q_1)$:

$s_2^* : q_2$ s.t. $q_2 \in \underset{q_2}{\operatorname{argmax}} \ q_2(\alpha - q_1 - q_2) - c q_2$

$$\Rightarrow q_2 = \boxed{\frac{1}{2}(\alpha - c - q_1)} = b_2(q_1)$$

$\underline{\Gamma(\phi)=\Gamma}$:



Player 1's best response: $s_1^*=q_1$ s.t.

$$q_1 \in \arg\max \tfrac{1}{2}\, q_1(\alpha-c-q_1)$$

$$\Rightarrow q_1^* = \tfrac{1}{2}(\alpha-c)$$

$$u_1(q_1, b_2(q_1)) = q_1\left(\alpha - q_1 - \tfrac{1}{2}(\alpha-c-q_1)\right) - cq_1$$
$$= \tfrac{1}{2}\, q_1(\alpha-c-q_1)$$

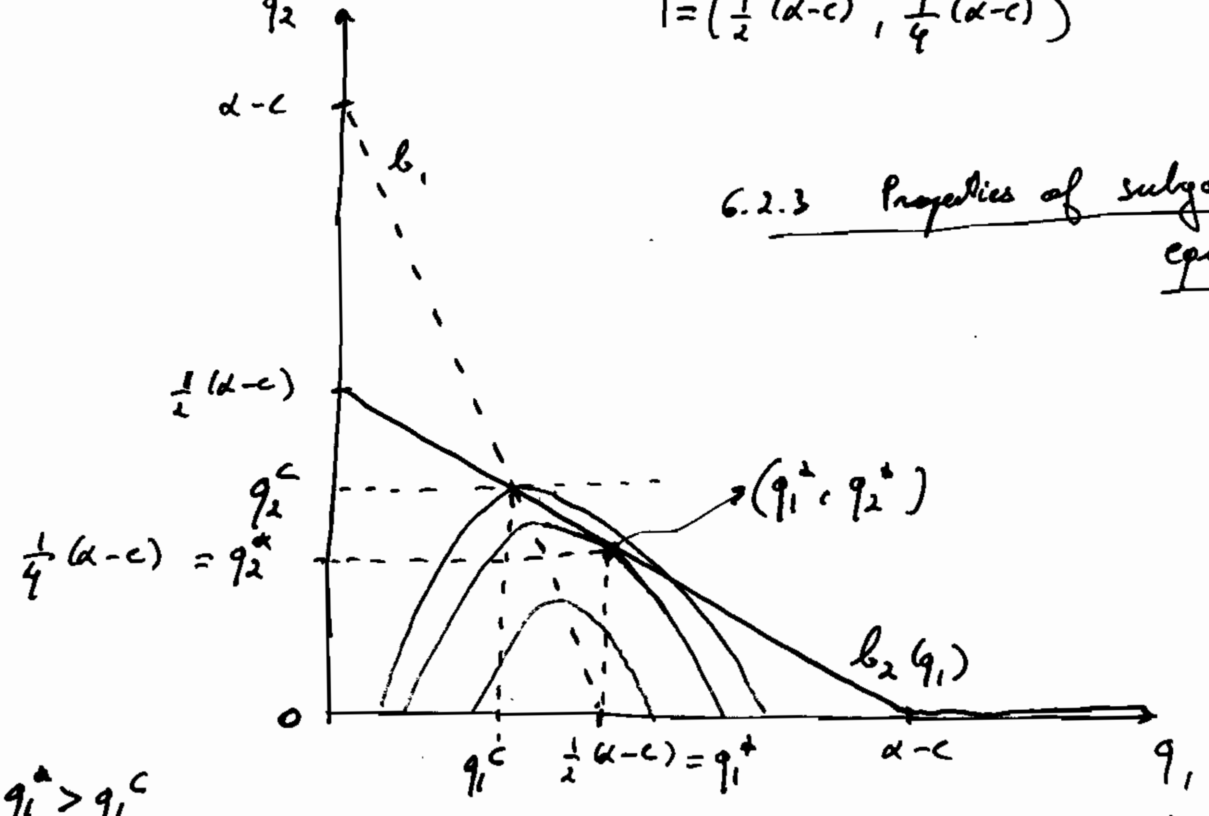
$\Rightarrow$ unique subgame perfect equilibrium (s_1^*, s_2^*) s.t.

$$\left[\begin{array}{l} s_1^*:\ q_1^* = \frac{1}{2}(\alpha-c) \\ s_2^*:\ s_2^*(q_1) = b_2(q_1) \\ \qquad\qquad = \frac{1}{2}(\alpha-c-q_1) \end{array}\right.$$

equilibrium outcome $O(s_1^*, s_2^*) = (q_1^*, b_2(q_1^*))$
$$= \left(\tfrac{1}{2}(\alpha-c),\ \tfrac{1}{4}(\alpha-c)\right)$$

$u_1(q_1^*, q_2^*) = \frac{1}{8}(\alpha-c)^2$

$u_2(q_1^*, q_2^*) = \frac{1}{16}(\alpha-c)^2$



6.2.3 Properties of subgame perfect equilibrium

$q_1^* > q_1^c$

$q_2^* < q_2^c$

Exercise 189.1

Exercise 191.1 $c_2(q_2) = \begin{cases} 0 & \text{if } q_2 = 0 \\ f + c\cdot q_2 & \text{if } q_2 > 0 \end{cases}$ $f > 0$

$\Gamma(q_1)$:

N.E. of $\Gamma(q_1)$:

$u_2(q_1,q_2) = q_2(\alpha - q_1 - q_2) - cq_2 - f$, if $q_2 > 0$
0 if $q_2 = 0$

s_2^*: $q_2 \in \arg\max_{q_2} u_2(q_1,q_2)$

$\Downarrow$

s_2^*: $q_2 = b_2(q_1)$ if $b_2(q_1)(\alpha - q_1 - b_2(q_1)) - cb_2(q_1) > f$

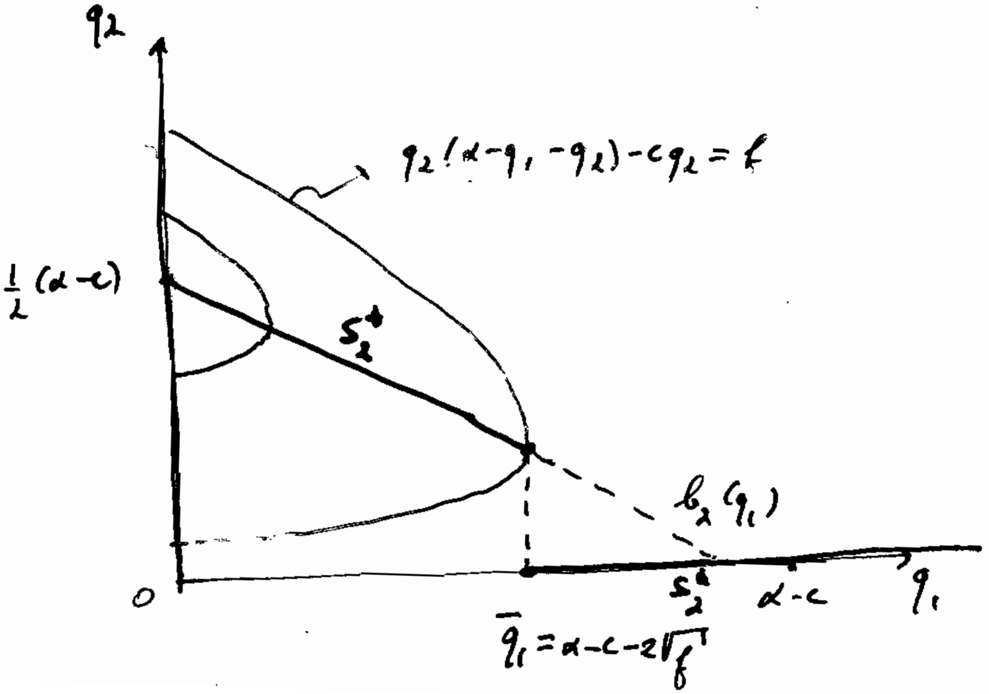
$= b_2(q_1)$ or 0 if ——— $= f$

$= 0$ if ——— $< f$

$b_2(q_1) = \frac{1}{2}(\alpha - c - q_1)$

$\Downarrow$ $b_2(q_1)(\alpha - q_1 - b_2(q_1)) - cb_2(q_1) = \frac{1}{4}(\alpha - c - q_1)^2$

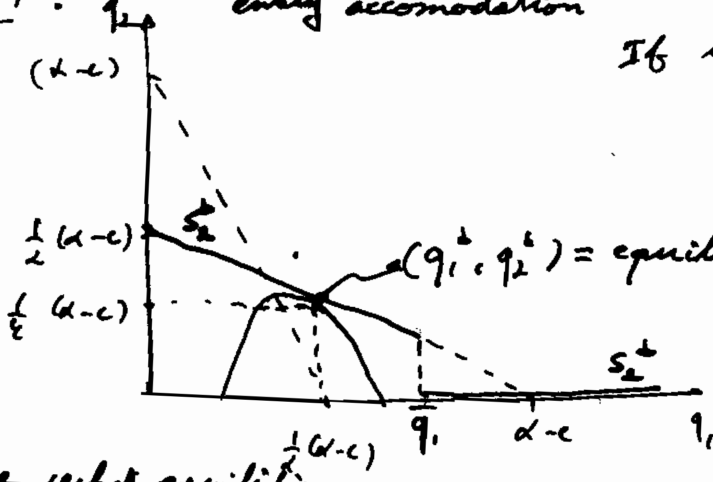
$\frac{1}{4}(\alpha - c - q_1)^2 \gtreqless f$ iff $q_1 \lesseqgtr \alpha - c - 2\sqrt{f} = \bar{q}_1$



$\Gamma(\phi)$

Case 1 : "entry accomodation"

If $u_1(\frac{1}{2}(\alpha-c), \frac{1}{4}(\alpha-c)) > u_1(\alpha-c-2\sqrt{f}, 0)$



(q_1^*, q_2^*) = equilibrium outcome

2 subgame perfect equilibria:

(s_1^*, s_2^*)

$s_1^* = q_1^* = \frac{1}{2}(\alpha-c)$

$s_2^* : \; s_2^*(q_1) = \frac{1}{2}(\alpha-c-q_1)$ if $q_1 \leq \alpha-c-2\sqrt{f}$

$= 0$ if $q_1 > \alpha-c-2\sqrt{f}$

(s_1^*, s_2^*)

$s_1^* = q_1^* = \frac{1}{2}(\alpha-c)$

$s_2^* : \; s_2^*(q_1) = \frac{1}{2}(\alpha-c-q_1)$ if $q_1 < \alpha-c-2\sqrt{f}$

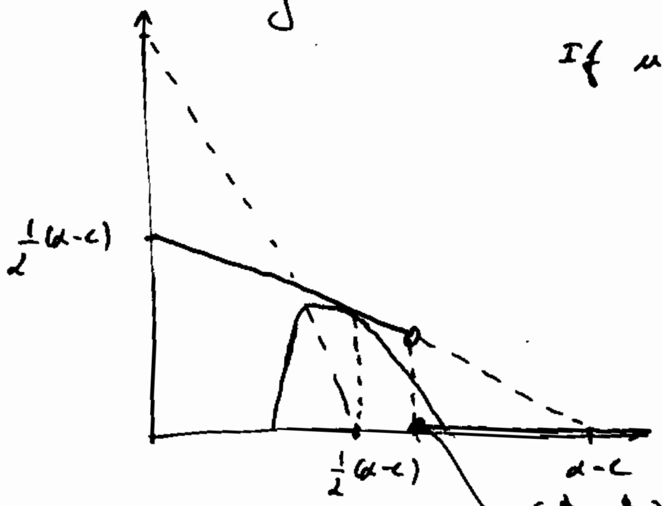
$= 0$ if $q_1 \geq \alpha-c-2\sqrt{f}$

case 2 : "entry deterrence"

If $u_1(\frac{1}{2}(\alpha-c), \frac{1}{4}(\alpha-c)) < u_1(\alpha-c-2\sqrt{f}, 0)$

example (exercise 191.1)

$c=0, \alpha=12, f=4$



$(q_1^*, q_2^*) = (\alpha-c-2\sqrt{f}, 0)$ equilibrium outcome

1 subgame perfect equilibrium.

(s_1^*, s_2^*)

$s_1^* = q_1^* = \alpha-c-2\sqrt{f}$

$s_2^* : \; s_2^*(q_1) = \frac{1}{2}(\alpha-c-q_1)$ if $q_1 < \alpha-c-2\sqrt{f}$

$= 0$ if $q_1 \geq \alpha-c-2\sqrt{f}$

(Case 3 : $u_1(\frac{1}{2}(\alpha-c), \frac{1}{4}(\alpha-c)) = u_1(\alpha-c-2\sqrt{f}, 0)$)

(Exercise 192.1)

Chap 7 Extensive games with perfect information: extensions

7.1 Allowing for simultaneous moves

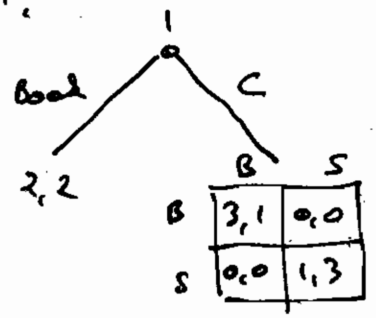
Definition 206.1 Extensive game with perfect information and simultaneous moves

• set of players	N
• set of terminal histories	Z
• player function	$P : \{\text{proper subhistories}\} \longrightarrow 2^N$ $h \longrightarrow P(h) \subseteq N$
• Sets of actions	if $i \in P(h)$: $A_i(h)$
• preferences over the set of terminal histories	$\succsim_i$ over Z $\hookrightarrow u_i$

Example 207.1 Variant of BoS

- $N = \{1,2\}$
- $Z = \{\text{Book}, (C,(B,B)), (C,(B,S)), (C,(S,B)), (C,(S,S))\}$
- $P(\emptyset) = 1$, $P(C) = \{1,2\}$
- $A_1(\emptyset) = \{\text{Book}, C\}$, $A_1(C) = \{B,S\}$, $A_2(C) = \{B,S\}$
- $u_1(C,(B,B)) = 3$, $u_1(\text{Book}) = 2$, $u_1(C,(S,S)) = 1$,
  $u_1(C,(B,S)) = u_1(C,(S,B)) = 0$

  $u_2(C,(S,S)) = 3$, $u_2(\text{Book}) = 2$, $u_2(C,(B,B)) = 1$,
  $u_2(C,(B,S)) = u_2(C,(S,B)) = 0$



Definitions: strategy of player i

outcome

Nash Equilibrium

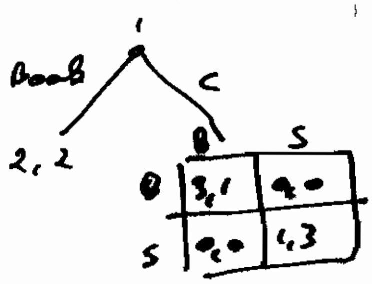
strategic form

subgame following history h

subgame perfect equilibrium

procedure of backward induction

Example $\overset{208.2}{210.1}$: Variant of BoS Γ



strategies of player 1: (Book, B), (Book, S)
(C, B), (C, S)

strategies of player 2: B, S

strategic form Γ^{st} :

	B	S
(Book, B)	2, 2	2, 2
(Book, S)	2, 2	2, 2
(C, B)	3, 1	0, 0
(C, S)	0, 0	1, 3

Nash equilibria: ((C, B), B), ((Book, B), S), ((Book, S), S)

Subgame $\Gamma(\phi) = \Gamma$

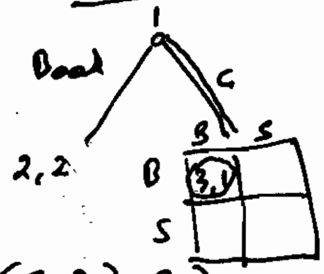
Subgame $\Gamma(C)$

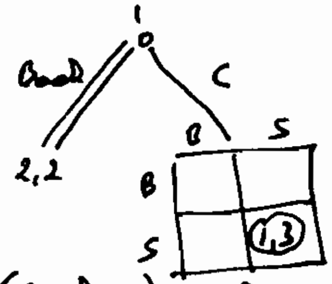
	B	S
B	3, 1	0, 0
S	0, 0	1, 3

Backward induction: $\Gamma(C)$: 2 NE (B, B), (S, S)

	B	S
B	3, 1	0, 0
S	0, 0	1, 3

$\Gamma(\phi) = \Gamma$:





subgame perfect equilibria: ((C, B), B) ((Book, S), S)

Exercise 210.2

Exercise 210.3

Exercise 211.1

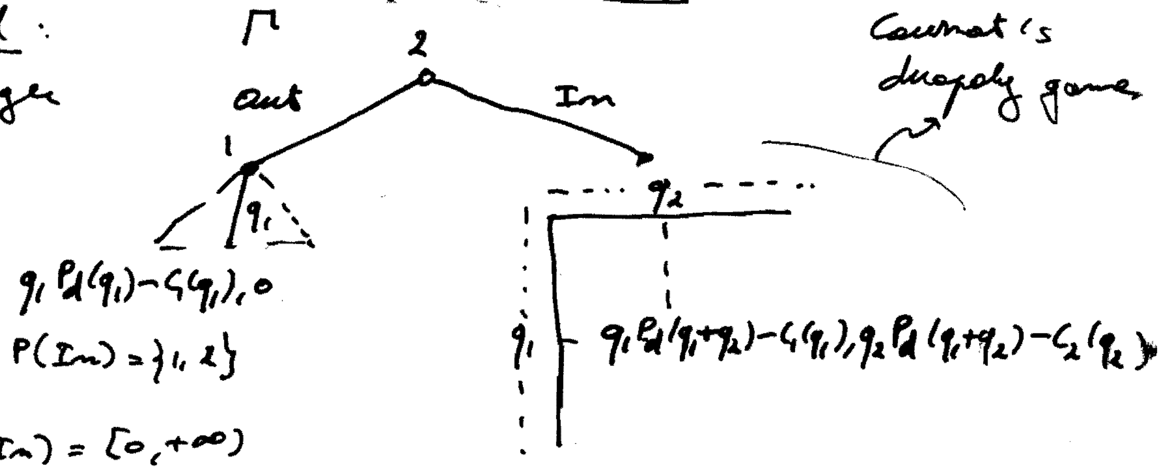
Exercise 211.2

(Exercise 212.1 G)

7.2 Illustration: entry into a monopolized industry

7.2.1 General Model:

- $N = \{1, 2\}$ → challenger
  ↳ incumbent
- $Z = \{(\text{Out}, q_1), (\text{In}, (q_1, q_2)) \mid q_1, q_2 \geq 0\}$
- $P(\emptyset) = \{2\}$, $P(\text{Out}) = \{1\}$, $P(\text{In}) = \{1, 2\}$
- $A_2(\emptyset) = \{\text{Out}, \text{In}\}$
  $A_1(\text{Out}) = A_2(\text{In}) = A_1(\text{In}) = [0, +\infty)$
- $u_1(\text{Out}, q_1) = q_1 P_d(q_1) - C_1(q_1)$, $u_2(\text{Out}, q_1) = 0$
  $u_1(\text{In}, (q_1, q_2)) = q_1 P_d(q_1+q_2) - C_1(q_1)$, $u_2(\text{In}, (q_1, q_2)) = q_2 P_d(q_1+q_2) - C_2(q_2) - f$



7.2.2 Example:

$C_1(q_1) = c.q_1$, $C_2(q_2) = c.q_2$. $P_d(Q) = \max(\alpha - Q, 0)$

$0 < c < \alpha$

Backward induction: I. $\Gamma(\text{In})$

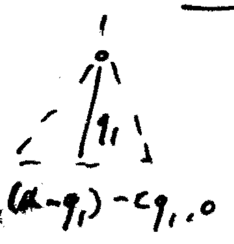
Cournot's duopoly game

→ only Nash equilibrium:

$$(q_1^*(\text{In}), q_2^*(\text{In})) = \left(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c)\right)$$

$$\begin{cases} u_1(O_{\text{In}}(q_1^*(\text{In}), q_2^*(\text{In})) = \frac{1}{9}(\alpha - c)^2 \\ u_2(O_{\text{In}}(q_1^*(\text{In}), q_2^*(\text{In}))) = \frac{1}{9}(\alpha - c)^2 - f \end{cases}$$

$\Gamma(\text{Out})$



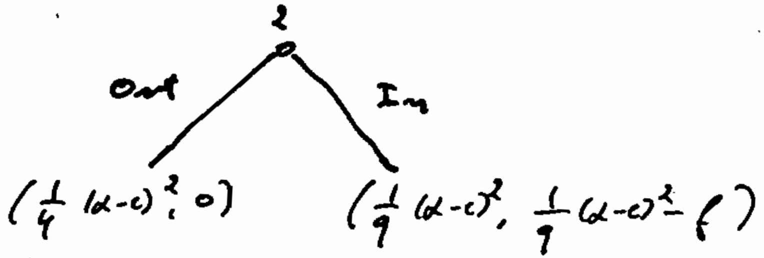
only Nash equilibrium:

$q_1^*(\text{Out}) =$ solut° of $\max\limits_{q_1} q_1(\alpha - q_1) - cq_1$

$$q_1^*(\text{Out}) = \frac{1}{2}(\alpha - c)$$

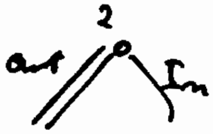
$$\begin{cases} u_1(O_{\text{Out}}(q_1^*(\text{Out}))) = \frac{1}{4}(\alpha - c)^2 \\ u_2(O_{\text{Out}}(q_1^*(\text{Out}))) = 0 \end{cases}$$

II. $\Gamma(f) = \Gamma$



Subgame perfect equilibria:

If $f > \frac{1}{9}(\alpha-c)^2$:



only one subgame perfect equilibrium:

$$\left(\left(\frac{1}{2}(\alpha-c), \frac{1}{3}(\alpha-c)\right), \left(\text{Out}, \frac{1}{3}(\alpha-c)\right)\right)$$

If $f < \frac{1}{9}(\alpha-c)^2$:

only one subgame perfect equilibrium:

$$\left(\left(\frac{1}{2}(\alpha-c), \frac{1}{3}(\alpha-c)\right), \left(\text{In}, \frac{1}{3}(\alpha-c)\right)\right)$$

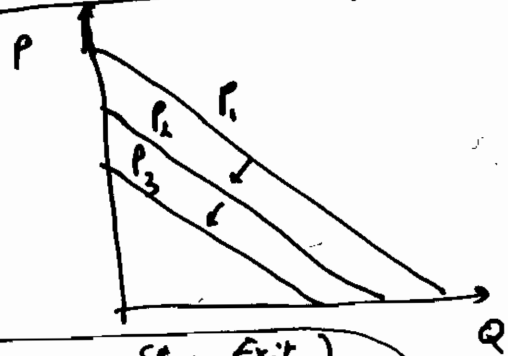
If $f = \frac{1}{9}(\alpha-c)^2$: 2 subgame perfect equilibria

$\left(\left(\frac{1}{2}(\alpha-c), \frac{1}{3}(\alpha-c)\right), \left(\text{Out}, \frac{1}{3}(\alpha-c)\right)\right)$; $\left(\left(\frac{1}{2}(\alpha-c), \frac{1}{3}(\alpha-c)\right), \left(\text{In}, \frac{1}{3}(\alpha-c)\right)\right)$

Exercise 214.1

(7.3 Electoral competition with strategic voters)
(7.4 Committee decision-making

7.5 Illustration: exit from a declining industry : $k_1 > k_2 > 0$



→ large firm
$N = \{1, 2\}$
→ small firm

erminal histories:

$((Stay, Stay), \dots, (Stay, Stay), (Exit, Exit))$

$((Stay, Stay), \dots, (Stay, Stay), (Stay, Exit), Stay, \dots, Stay, Exit)$ → Firm 2 exits at period r → Firm 1 exits at period t

$1 \quad \dots \quad r-1 \quad r \quad r+1 \dots t-1 \quad t$

$((Stay, Stay), \dots, (Stay, Stay), (Exit, Stay), Stay, \dots, Stay, Exit)$ → Firm 2 exits at period t

↓ Firm 1 exits at period r

$P(\emptyset) = P((Stay, Stay), \dots, (Stay, Stay)) = \{1, 2\}$

$P(\dots, (Stay, Exit), \dots) = \{1\}$, $P(\dots, (Exit, Stay), \dots) = \{2\}$

$A_i(h) = \{1, 2\}$, for all i, h s.t. $i \in P(h)$

$u_1(\underset{1}{(Stay, Stay)}, \dots, \underset{r-1}{(Stay, Stay)}, \underset{r}{(Exit, Exit)}) =$

$u_1((Stay, Stay), \dots, (Stay, Stay), (Exit, Stay), Stay, \dots, Stay, Exit)$

$\sum_{s=1}^{r-1} (P_s(k_1 + k_2) - c) . k_1$ → Firm 1's (fixed) production → Firm 2's (fixed) production

$u_1(\underset{1}{(Stay, Stay)}, \dots, \underset{r-1}{(Stay, Stay)}, \underset{r}{(Stay, Exit)}, \underset{r+1}{Stay}, \dots, \underset{t-1}{Stay}, \underset{t}{Exit}) =$

$\sum_{s=1}^{r-1} (P_s(k_1 + k_2) - c) k_1 + \sum_{s=r}^{t-1} (P_s(k_1) - c) k_1$ → unit cost

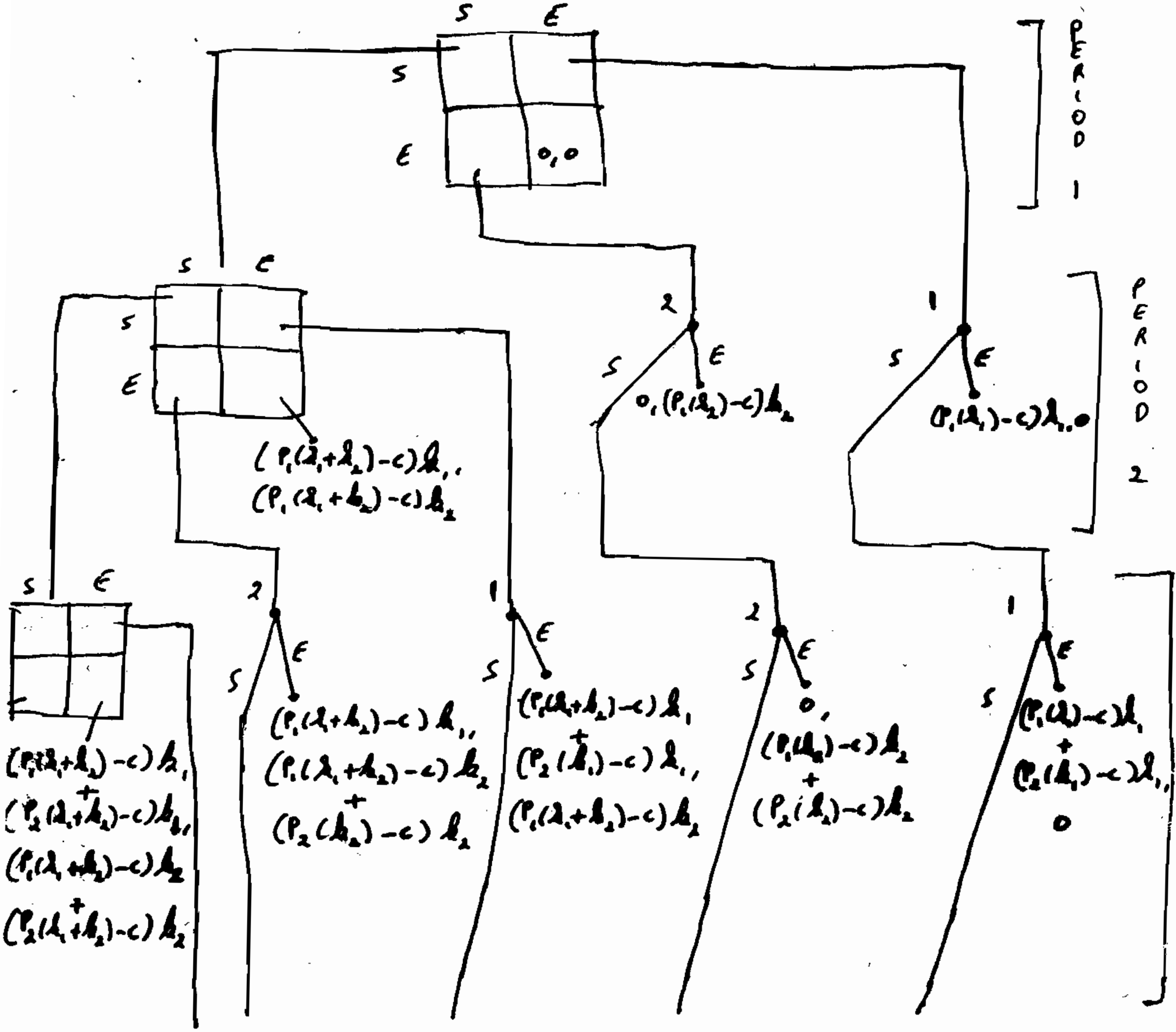
$u_2(\underset{1}{(Stay, Stay)}, \dots, \underset{r-1}{(Stay, Stay)}, \underset{r}{(Exit, Exit)}) =$

$u_2((Stay, Stay), \dots, (Stay, Stay), (Stay, Exit), Stay, \dots, Stay, Exit)$

$\sum_{s=1}^{r-1} (P_s(k_1 + k_2) - c) k_2$

$u_2(\underset{1}{(Stay, Stay)}, \dots, \underset{r-1}{(Stay, Stay)}, \underset{r}{(Exit, Stay)}, \underset{r+1}{Stay}, \dots, \underset{t-1}{Stay}, \underset{t}{Exit})$

$\sum_{s=1}^{r-1} (P_s(k_1 + k_2) - c) . k_2 + \sum_{s=r}^{t-1} (P_s(k_2) - c) . k_2$


S
E
S
E
0, 0
PERIOD 1
S
E
S
E
(P1(λ1+λ2)-c)λ1,
(P1(λ1+λ2)-c)λ2
2
S
E
0, (P1(λ2)-c)λ2
1
S
E
(P1(λ1)-c)λ1, 0
PERIOD 2
S
E
(P1(λ1+λ2)-c)λ1
+
(P2(λ1+λ2)-c)λ1,
(P1(λ1+λ2)-c)λ2
+
(P2(λ1+λ2)-c)λ2
2
S
E
(P1(λ1+λ2)-c)λ1,
(P1(λ1+λ2)-c)λ2
+
(P2(λ2)-c)λ2
1
S
E
(P1(λ1+λ2)-c)λ1
+
(P2(λ1)-c)λ1,
(P1(λ1+λ2)-c)λ2
2
S
E
0,
(P1(λ2)-c)λ2
+
(P2(λ2)-c)λ2
1
S
E
(P1(λ1)-c)λ1
+
(P2(λ1)-c)λ1,
0
PERIOD 3

~~Example~~
Exercise 224.1

$c = 10, \quad k_1 = 40, \quad k_2 = 20$

$P_t(Q) = \max(100.5 - t - Q, 0)$

Let t_i be the last period where Firm i, if alone in the market, is profitable.

$$t_1 = \max\{t \mid \underbrace{P_t(k_1)}_{\shortparallel} \geq c\}$$

$$100.5 - t - k_1 \geq c \iff t \leq 100.5 - k_1 - c = 50.5$$

$$\boxed{t_1 = 50}$$

$$t_2 = \max\{t \mid \underbrace{P_t(k_2)}_{\shortparallel} \geq c\}$$

$$100.5 - t - k_2 \geq c \iff t \leq 100.5 - k_2 - c = 70.5$$

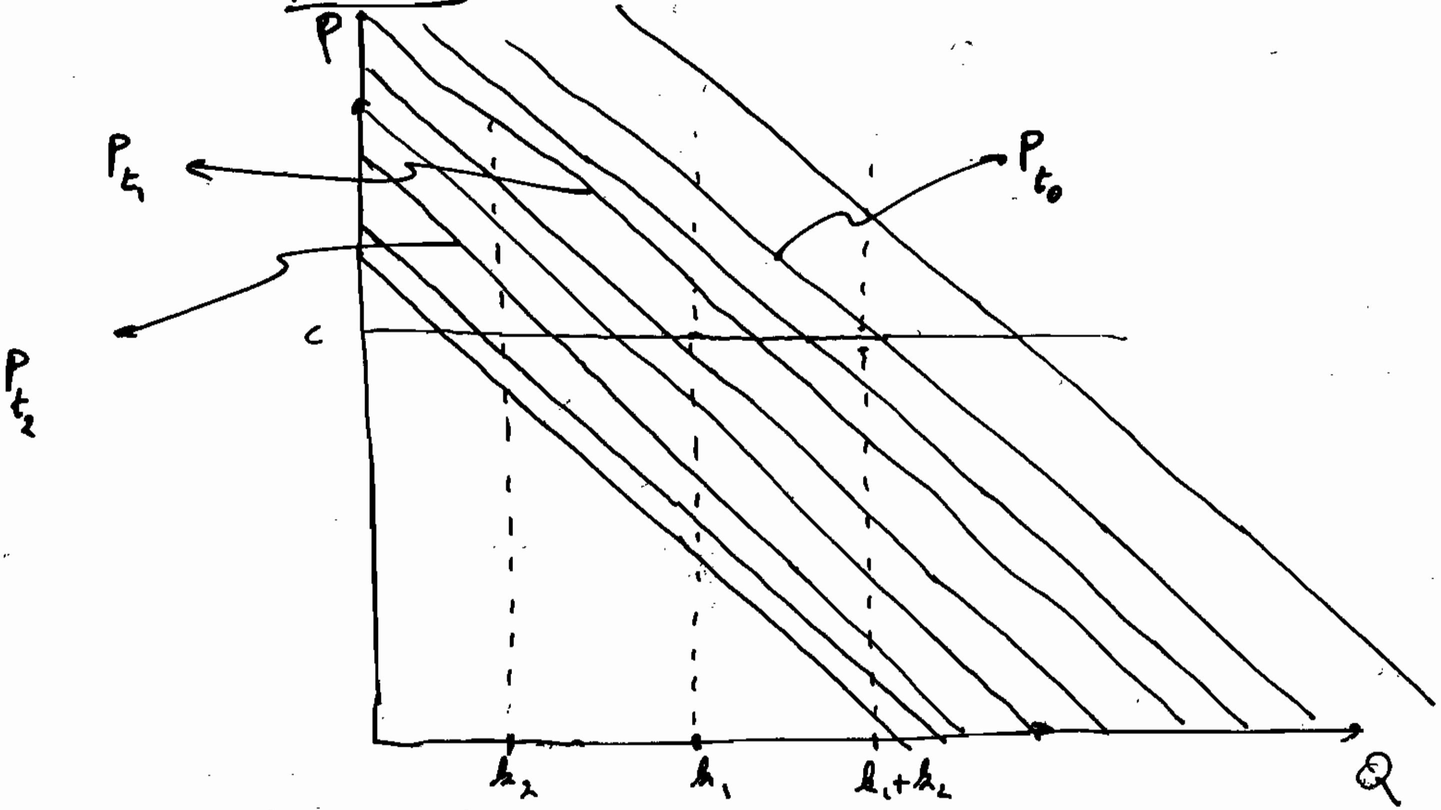
$$\boxed{t_2 = 70}$$

Let t_0 be the last period where Firms 1 and 2, if both in the market, are profitable.

$$t_0 = \max\{t \mid \underbrace{P_t(k_1 + k_2)}_{\shortparallel} \geq c\}$$

$$100.5 - t - k_1 - k_2 \geq c \iff t \leq 100.5 - k_1 - k_2 - c = 30.5$$

$$\boxed{t_0 = 30}$$



$t_0 = 30 < t_1 = 50 < t_2 = 70$

Subgame perfect equilibrium :

For all subgame $\Gamma_t(h)$ starting at Period $t > t_2 = 70$:

Every period a firm stays in, its profit < 0

$$\Rightarrow \left| \begin{array}{l} s_1^*|_{\Gamma_t(h)} : \text{Exit} \\ s_2^*|_{\Gamma_t(h)} : \text{Exit} \end{array} \right.$$

For all subgame $\Gamma_t(h)$ starting at Period t s.t. $t_1 = 50 < t \leq t_2 = 70$

Every period Firm 1 stays in, its profit < 0

$$\Rightarrow s_1^*|_{\Gamma_t(h)} : \text{Exit}$$

$\Rightarrow$ As long as Firm 2 stays in, its profit > 0

$$\Rightarrow s_2^*|_{\Gamma_t(h)} : \left(\begin{array}{l} \text{Stay until Period } 71 \\ \text{Exit at Period } 71 \end{array} \right.$$

– subgame $\Gamma_{t_1}(h) = \Gamma_{t_1}((\text{Stay}, \text{Stay}), ---, (\text{Stay}, \text{Stay}))$ starting at Period $t_1 = 50$

		Firm 2 at Period $t_1=50$	
		S	E
Firm 1 : Period $t_1=50$	S	$\left(\underbrace{(P_{t_1}(k_1+k_2)-c)k_1}_{\overset{\wedge}{0}}, \boxed{(P_{t_1}(k_1+k_2)-c)k_2 + \sum_{t=t_1+1}^{t_2}(P_t(k_2)-c)k_2}\right)$	$\left(\underbrace{(P_{t_1}(k_1)-c_1)k_1}_{\overset{\vee}{0}}, 0\right)$
	E	$\wedge$ $\left(0, \boxed{\sum_{t=t_1}^{t_2}(P_t(k_2)-c)k_2}\right)$	$(0 , 0)$

$$\underbrace{(P_{t_1}(k_1+k_2)-c)k_2}_{\overset{\wedge}{0}} + \underbrace{\sum_{t=t_1+1}^{t_2}(P_t(k_2)-c)k_2}_{\overset{\vee}{0}} = (\max(100.5-50-40-20, 0)-10).20$$

$$+$$

$$\underbrace{((100.5-51-20)-10)}_{\overset{\shortparallel}{19.5}}20 + \cdots + \underbrace{((100.5-70-20)-10)}_{\overset{\shortparallel}{0.5}}20$$

$$= -200 + \frac{20.20}{2}.20 = -200 + 4000 \boxed{> 0}$$

$$\Rightarrow s_1^*|_{\Gamma_t(h)} : \text{Exit} \quad , \quad s_2^*|_{\Gamma_t(h)} : \left(\begin{array}{l} \text{Stay until Period } 71 \\ \text{Exit at Period } 71 \end{array} \right.$$

For subgame $\Gamma_{t_1-1}(h)=\Gamma_{t_1-1}((Stay,Stay),\dots,(Stay,Stay))$ starting at Period $t_1-1=49>t_0=30$

Firm 2 at Period t_1-1

Firm 1 at Period t_1-1

$$\begin{array}{c|cc}
 & S & E \\
\hline
S & \left(\underbrace{(P_{t_1-1}(\ell_1+\ell_2)-c)\ell_1}_{\overset{\wedge}{0}},\ (P_{t_1-1}(\ell_1+\ell_2)-c)\ell_2+\sum_{t=t_1}^{t_2}(P_t(\ell_2)-c)\ell_2\right) & \left(\underbrace{(P_{t_1-1}(\ell_1)-c)\ell_1}_{\overset{\vee}{0}},\ 0\right) \\
 & \wedge & \\
E & \left(0,\ \sum_{t=t_1-1}^{t_2}(P_t(\ell_2)-c)\ell_2\right) & (0,\ 0)
\end{array}$$

$$\underbrace{P_{t_1-1}(\ell_1+\ell_2)-c)\ell_2}_{\overset{\wedge}{0}}+\underbrace{\sum_{t=t_1}^{t_2}(P_t(\ell_2)-c)\ell_2}_{\overset{\vee}{0}} \geqslant (P_{t_1}(\ell_1+\ell_2)-c)\ell_2+\sum_{t=t_1+1}^{t_2}(P_t(\ell_2)-c)\ell_2=-200+4000$$

>0

$\Rightarrow$ $s_1^b|_{\Gamma_t(h)}=$ Exit

$s_2^b|_{\Gamma_t(h)}:\begin{cases}\text{Stay until Period 71}\\ \text{Exit at Period 71}\end{cases}$

⋮

$\Rightarrow$ For subgame $\Gamma_t(h)=\Gamma_t((Stay,Stay),\dots,(Stay,Stay))$ starting at Period t s.t. $t_0=30<t\leq t_1=50$

$s_1^b|_{\Gamma_t(h)}$: Exit

$s_2^b|_{\Gamma_t(h)}:\begin{cases}\text{Stay until Period 71}\\ \text{Exit at Period 71}\end{cases}$

For all subgame $\Gamma_t(h)$ starting at Period $t \le t_0 = 30$

every period $t' \le t_0 = 30$ a firm stays in, its profit > 0

$$\Rightarrow \quad s_1^*|_{\Gamma_t(h)} : \begin{cases} \text{Stay until Period } t_0+1 = 31 \\ \text{Exit at Period } 31 \end{cases}$$

$$s_2^*|_{\Gamma_t(h)} : \begin{cases} \text{Stay until Period } t_2+1 = 71 \\ \text{Exit at Period } 71 \end{cases}$$

$\Rightarrow$ only one subgame perfect equilibrium (s_1^*, s_2^*)

s_1^* : Following history h at Period t: $\longrightarrow$ h does not contain (Exit, Stay) (Exit, Exit)

$$\begin{cases} \text{Stay} & \text{if } t \le t_0 = 30 \\ \text{Stay} & \text{if } t_0 < t \le t_1^{=50} \text{ and (Stay, Exit) is part of } h \\ \text{Exit} & \text{if } t_0 < t \le t_1 \text{ and } h = ((\text{Stay}, \text{Stay}), \dots, (\text{Stay}, \text{Stay})) \\ \text{Exit} & \text{if } t > t_1 \end{cases}$$

s_2^* : Following history h at Period t $\longrightarrow$ h does not contain (Stay, Exit) (Exit, Exit)

$$\begin{cases} \text{Stay} & \text{if } t \le t_2 = 70 \\ \text{Exit} & \text{if } t > t_2 \end{cases}$$

Outcome of (s_1^*, s_2^*) : $\begin{cases} \text{Firm 1 exits at } t_0+1 = 31 \\ \text{Firm 2 exits at } t_2+1 = 71 \end{cases}$

Effect of a constraint on firm 2's debt

exercise 22.5.1

If Firm 2's debt cannot exceed $\bar{d} \geq 0$: New payoff function $\tilde{u}_2$ for Firm 2

$\tilde{u}_2(h) =$

If, for all periods $t < t'$, $\sum_{s=t}^{t'}$ Firm 2's profit$_s(h) \geq -\bar{d}$: $\tilde{u}_2(h) = u_2(h)$

If there exists $t < t'$ s.t. $\sum_{s=t}^{t'}$ Firm 2's Profit$_s(h) < -\bar{d}$: $\tilde{u}_2(h) = -\#S(h).K$

where $\#S(h) = \#$ of periods Firm 2 stayed in h

K sth. positive and very large

(or as soon as Firm 2's cumulated debt is higher than $\bar{d}$, Firm 2 is forced out and cannot come back)

If $\bar{d} < 810$, then there exists no subgame perfect equilibrium where (as in (S_1^*, S_2^*))
| Firm 1 exits at Period $t_0 + 1 = 31$

If there existed such an equilibrium $(\tilde{S}_1^*, \tilde{S}_2^*)$, there would exist a strictly profitable deviation by Firm 1 in $\Gamma_{31}((\text{Stay}, \text{Stay}), \dots, (\text{Stay}, \text{Stay})) = \Gamma_{31}(h)$

In every subgame $\Gamma_t(\bar{h})$ s.t. Firm 2's cumulated debt $> \bar{d}$: $\tilde{S}_2^*$: Exit

$\bar{S}_1^*$: Stay from Period 31 to Period $t_1 = 50$
Exit at Period $t_1 + 1 = 51$

In particular $\Gamma_{39}((\text{Stay}, \text{Stay}), \dots, (\text{Stay}, \text{Stay}))$

$$\sum_{s=31}^{39} \text{Firm 2's Profit}_s = (P_{31}(k_1+k_2)-c)k_2 + \dots + (P_{39}(k_1+k_2)-c)k_2$$

$$= (\underbrace{\underbrace{100.5-31-60}_{9.5}-10}_{-0.5})20 + \dots + (\underbrace{\underbrace{100.5-39-60}_{1.5}-10}_{-8.5})20$$

$$= -(0.5 + 1.5 + \dots + 8.5)20$$

$$= -\frac{9.9}{2}\,20 = -810 < -\bar{d}$$

$$\sum_{s=31}^{50} \text{Firm 1's Profit}_s(\bar{s}_1^*, \tilde{s}_2^*) \geq \sum_{s=31}^{50} \text{Firm 1's Profit}_s(\bar{s}_1^*, \text{Firm 2 stays until } t=39)$$

$$= (P_{31}(l_1+l_2)-c)\,l_1 + \cdots + (P_{39}(l_1+l_2)-c)\,l_1$$

$$+ (P_{40}(l_1)-c)\,l_1 + \cdots + (P_{50}(l_1)-c)\,l_1$$

$$= -(0.5+1.5+\cdots+8.5)\,40$$

$$+ \underbrace{(\underbrace{100.5-40-40}_{20.5}-10)}_{10.5}.40 + \cdots + \underbrace{(\underbrace{100.5-50-40}_{10.5}-10)}_{0.5}.40$$

$$= -\frac{9.9}{2}.40 + \underbrace{(10.5+\cdots+0.5)}_{\overset{\shortparallel}{\frac{11.11}{2}}}\,40 > 0$$

Chap 8 Coalitional Games and the Core

Definition 239.1 : Coalitional Game

- Set of players — N
- for each coalition, a set of actions — for each $C \subseteq N, C \neq \phi$: $A(C)$
- for each player, preferences over the set of actions of all coalitions including this player — $\succsim_i$ over $\bigcup_{C \ni i} A(C)$ $\hookrightarrow u_i$

Example 240.1 : Two-player unanimity game

- $N = \{1, 2\}$
- $A(\{1\}) = A(\{2\}) = \{(0,0)\}$

  $A(\{1,2\}) = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, \text{ and } x_1 + x_2 = 1\}$ — set of $\{1,2\}$-allocations of 1 unit
- $u_i((x_1, x_2)) = x_i$

Example: Landowner and workers with production function f

f increasing and s.t. $f(0) = 0$

- $N = \{\overset{\text{landowner}}{1}, \underbrace{2, \dots, m+1}_{\text{workers}}\}$
- $A(C) = \{(0, \dots, 0)\}$

  $A(C) = \{C\text{-allocations of } f(\#C)\}$, for all C s.t. $1 \in C$
- $u_i((x_1, \dots, x_{m+1})) = x_i$

<u>Example 240.3</u> Three-player majority game

- $N = \{1, 2, 3\}$
- $A(\{1\}) = A(\{2\}) = A(\{3\}) = \{(0,0,0)\}$

  $A(C) = \{C\text{-allocations of } 1 \text{ unit}\}$, for all C s.t. $\#C \geq 2$
- $u_i((x_1, x_2, x_3)) = x_i$

<u>Definition</u>: Coalitional game with transferable payoff

(N, A, u_i) has transferable payoff iff there exists $v : \{C \mid C \subseteq N, C \neq \emptyset\} \to \mathbb{R}$ s.t.

$A(C) = \{C\text{-allocations of } v(C)\}$, for all $C \neq \emptyset$, $C \subseteq N$

and

$u_i((x_1, \dots, x_n)) = x_i$, for all i

<u>Examples</u>:

Two-player unanimity game: $v(\{1\}) = v(\{2\}) = 0$, $v(\{1,2\}) = 1$

Landowner-worker game: $v(C) = 0$ if $1 \notin S$

$= f(\#C)$ if $1 \in S$

Three-player majority game: $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

$v(C) = 1$, if $\#C \geq 2$

<u>Exercise 241.2</u>

<u>Example 242.1</u> : House allocation game with valuations $\boxed{v_1^i}, \dots, v_n^i$, $1 \leq i \leq n$

player i's valuation of player 1's house.

$N = \{1, \dots, n\}$

$A(C) = \{(x_1, \dots, x_n) \mid x_1 = 0$, for all $i \notin C$

and there exists

$f : C \to C$ s.t.

$f(i) \neq f(j)$, for all $i \neq j$

$x_{f(i)} = v_i^{f(i)}$, for all $i \in C\}$

$u_i((x_1, \dots, x_n)) = x_i$

payoff is not transferable

Example 242.2 Marriage market with valuations $(v_1, \dots, v_m)$

$(v_{m+1}, \dots, v_{m+w})$

$v_1^i, \dots, v_m^i$, $m+1 \le i \le m+w$

$v_{m+1}^i, \dots, v_{m+w}^i$, $1 \le i \le m$

$$N = \{\overbrace{1, \dots, m}^{\text{men}}, \underbrace{m+1, \dots, m+w}_{\text{women}}\}$$

$$A(C) = \{ \{S_1, \dots, S_p\} \mid S_i \cap S_j = \phi, \text{ for all } i \ne j \in C$$
$$\# S_i = 1 \text{ or } 2, \text{ for all } i \in C$$
$$S_1 \cup \dots \cup S_p = C$$
$$\text{if } \# S_i = 2, \text{ there exist } 1 \le k \le m,\ m+1 \le l \le m+w \text{ s.t. } S_i = \{k, l\} \}$$

player i's valuation of staying single

Woman i's valuation of man m

$$u_i(\{S_1, \dots, S_p\}) = 0 \quad \text{if } i \notin \bigcup_{l=1}^{p} S_l$$
$$= v_i \quad \text{if there exists } l \text{ s.t. } S_l = \{i\}$$
$$= v_j^i \quad \text{if there exists } l \text{ s.t. } S_l = \{i, j\}$$

payoff is not transferable

Definition 242.3 Cohesive coalitional game

Coalitional game (N, A, u_i) is cohesive iff

$$\text{for all } \{S_1, \dots, S_k\},\ \{a_{S_1}, \dots, a_{S_k}\} \text{ s.t. } S_1 \cup \dots \cup S_k = N,\ a_{S_i} \in S_i, \text{ for } 1 \le i \le k$$

$$\text{there exists } a_N \in A(N) \text{ s.t. } u_i(a_{S_j}) \le u_i(a_N), \text{ for all } i, j \text{ s.t. } i \in S_j$$

Coalitional game with transferable payoff (N, v) is cohesive

iff

$$v(S_1) + \dots + v(S_k) \le v(N), \text{ for all } \{S_1, \dots, S_k\} \text{ s.t.}$$
$$S_1 \cup \dots \cup S_k = N$$
$$S_i \cap S_j = \phi, \text{ for all } i \ne j$$
$$S_i \ne \phi, \text{ for all } i$$

Examples: Two-player unanimity game, Landowner-worker game, Three-player majority game, House-allocation game, Marriage market game, Exercise 243.1

8.2 The core

Definition 243.2

- Core of a coalitional game

$\text{Core}(N, A, u_i) = \{ a_N \mid a_N \in A(N)$ and there exist no C, $a \in A(C)$ s.t. $u_i(a_N) < u_i(a)$, for all $i \in C \}$

- core of a coalitional game with transferable payoff

$\text{Core}(N, v) = \{$ N-allocation $(x_1, \dots, x_n)$ of $v(N) \mid \sum_{i \in C} x_i \geq v(C)$, for all $C \}$

Example 244.1: 2-player unanimity game

$\text{Core} = \{ (x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1 \} \quad (= A(\{1,2\}))$

- Variant of 2-player unanimity game

$v(\{1\}) = p, \quad v(\{2\}) = q, \quad v(\{1,2\}) = 1$

$\text{Core} = \{ (x_1, x_2) \mid x_1 \geq p, x_2 \geq q, x_1 + x_2 = 1 \}$

Example 244.2: Landowner-worker game with 2 workers

$\text{Core} = \{ (x_1, x_2, x_3) \mid x_1 \geq f(1)$

$x_2 + x_3 \leq f(3) - f(1) \Leftarrow$

$x_2 \geq 0$

$x_3 \geq 0$

$x_3 \leq f(3) - f(2) \Leftrightarrow x_1 + x_2 \geq f(2)$

$x_2 \leq f(3) - f(2) \Leftrightarrow x_1 + x_3 \geq f(2)$

$x_1 = f(3) - x_2 - x_3 \leftarrow x_1 + x_2 + x_3 = f(3) \}$

$\Rightarrow \text{Core} = \{ (x_1, x_2, x_3) \mid$

$0 \leq x_2 \leq f(3) - f(2)$

$0 \leq x_3 \leq f(3) - f(2)$

$x_2 + x_3 \leq f(3) - f(1)$

$x_1 + x_2 + x_3 = f(3) \}$

Exercise 245.1

Exercise 245.2

Exercise 245.3

Exercise 245.4

Example 245.5 Market with one owner and two homogeneous buyers
with the same valuation 1

$N = \{\overset{\text{owner}}{1}, \underbrace{2, 3}_{\text{(potential) buyers}}\}$

$A(\{2,3\}) = \{(1,0,0;0,x_2,x_3) \mid x_2, x_3 \geq 0,\ x_2 + x_3 = \overset{\text{buyer 2's wealth}}{m_2} + \underset{\text{buyer 3's wealth}}{m_3}\}$ ≥ 1

$A(\{1,2,3\}) = \{(\underbrace{y_1, y_2, y_3}_{\{1,2,3\}\text{-allocation of the good}}; \underbrace{x_1, x_2, x_3}_{\{1,2,3\}\text{-allocation of money}}) \mid x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 = m_2 + m_3$
$(y_1, y_2, y_3) = (1,0,0)$ or $(0,1,0)$ or $(0,0,1)\}$

$A(\{1,2\}) = \{(y_1, y_2, 0; x_1, x_2, m_3) \mid x_1, x_2 \geq 0,\ x_1 + x_2 = m_2$
$(y_1, y_2) = (1,0)$ or $(0,1)\}$

$A(\{1,3\}) = \{(y_1, 0, y_3; x_1, m_2, x_3) \mid x_1, x_3 \geq 0,\ x_1 + x_3 = m_3$
$(y_1, y_3) = (1,0)$ or $(0,1)\}$

$A(\{1\}) = A(\{2\}) = A(\{3\}) = \{(1,0,0;0,m_2,m_3)\}$

$$\begin{aligned} u_i((y_1, y_2, y_3; x_1, x_2, x_3)) &= y_i + x_i, \text{ if } i = 2 \text{ or } 3 \\ &= x_i, \text{ if } i = 1 \end{aligned}$$

$\rightarrow$ Core $= \{(0,1,0;1,m_2-1,m_3),\ (0,0,1;1,m_2,m_3-1)\}$

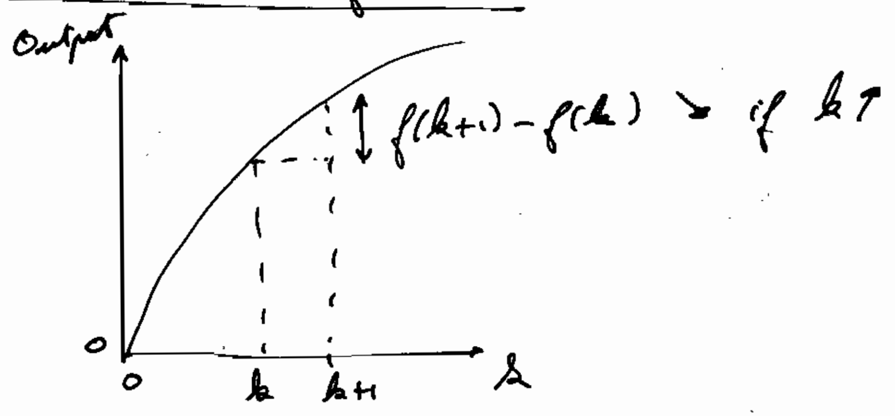
Exercise 246.1

(Exercise 247.1)

8.3 Illustration: ownership and the distribution of wealth

Production function

$n \geq 3$



8.3.1 Single landowner and $\underbrace{\text{landless}}_{n-1}$ workers

$$\begin{aligned} v(C) &= f(\#C), \text{ if } 1 \in C \\ &= 0, \text{ if } 1 \notin C \end{aligned}$$

Core $= \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = f(n),\ 0 \leq x_2, \ldots, x_n \leq f(n) - f(n-1)\}$

Exercise 248.1

core and competitive equilibrium.

Assume $(y_1, y_2, y_3; x_1, x_2, x_3) \in$ Core

- If $y_2 = 0 \Rightarrow x_2 \geq m_2$

Otherwise

$$u_2((y_1, y_2, y_3; x_1, x_2, x_3)) = x_2 < m_2 = u_2((1,0,0;0,m_2,m_3))$$

and $(1,0,0;0,m_2,m_3) \in A(\{2\})$

$\Rightarrow \{2\}$ can impose upon $(y_1, y_2, y_3; x_1, x_2, x_3)$

$\downarrow$

Impossible

- If $y_2 = 0 \Rightarrow x_2 \leq m_2$

Otherwise $x_2 > m_2$

$$((y_1, y_2, y_3; x_1, x_2, x_3)) = x_1 < x_1 + \frac{x_2 - m_2}{2} = u_1((y_1, y_2, y_3; x_1 + \frac{x_2 - m_2}{2}, m_2, x_3 + \frac{x_2 - m_2}{2}))$$

$$_3((y_1, y_2, y_3; x_1, x_2, x_3)) = y_3 + x_3 < y_3 + x_3 + \frac{x_2 - m_2}{2} = u_3((y_1, y_2, y_3; x_1 + \frac{x_2 - m_2}{2}, m_2, x_3 + \frac{x_2 - m_2}{2}))$$

and $(y_1, y_2, y_3; x_1 + \frac{x_2 - m_2}{2}, m_2, x_3 + \frac{x_2 - m_2}{2}) \in A(\{1,3\})$

$\Rightarrow \{1,3\}$ can impose upon $(y_1, y_2, y_3; x_1, x_2, x_3)$

$\downarrow$

Impossible

$\Rightarrow$ • If $y_2 = 0 \Rightarrow x_2 = m_2$

[Similarly :

• If $y_3 = 0 \Rightarrow x_3 = m_3$

- If $y_2 = 1 \Rightarrow x_2 \geq m_2 - 1$

Otherwise

$$u_2((y_1, y_2, y_3; x_1, x_2, x_3)) = 1 + x_2 < m_2 = u_2((1,0,0;0,m_2,m_3))$$

and

$$(1,0,0;0,m_2,m_3) \in A_2(\{2\})$$

$\Rightarrow \{2\}$ can impose upon $(y_1, y_2, y_3; x_1, x_2, x_3)$

$\downarrow$

Impossible

- If $y_2 = 1 \Rightarrow x_2 \leq m_2 - 1$ Otherwise $x_2 > m_2 - 1$

$$u_1((\underset{0}{\underset{\shortparallel}{\underbrace{y_1}}}, \underset{1}{\underset{\shortparallel}{\underbrace{y_2}}}, \underset{0}{\underset{\shortparallel}{\underbrace{y_3}}}; x_1, x_2, \underset{m_3}{\underset{\shortparallel}{\underbrace{x_3}}})) = x_1 = m_2 - x_2 < \frac{(m_2 - x_2) + 1}{2} = u_1((0,0,1; \frac{(m_2 - x_2)+1}{2}, m_2, m_3 - \frac{(m_2 - x_2)+1}{2}))$$

$$_3((y_1, y_2, y_3; x_1, x_2, x_3)) = m_3 < \underbrace{1 + m_3 - \frac{(m_2 - x_2)+1}{2}}_{\hookrightarrow > 0} = u_3((0,0,1; \frac{(m_2 - x_2)+1}{2}, m_2, m_3 - \frac{(m_2 - x_2)+1}{2}))$$

and $(0,0,1; \frac{(m_2 - x_2)+1}{2}, m_2, m_3 - \frac{(m_2 - x_2)+1}{2}) \in A(\{1,3\})$

$\Rightarrow \{1,3\}$ can impose upon $(y_1, y_2, y_3; x_1, x_2, x_3)$

$\downarrow$

Impossible

$\Rightarrow$ • If $y_2 = 1 \Rightarrow x_2 = m_2 - 1$

Similarly:

- If $y_3 = 1 \Rightarrow x_3 = m_3 - 1$

- $y_1 = 0$. Otherwise

$$(y_1, y_2, y_3; x_1, x_2, x_3) = (1,0,0;0,m_2,m_3)$$

$$u_1((y_1, y_2, y_3; x_1, x_2, x_3)) = 0 < \frac{1}{2} = u_1((0,1,0; \frac{1}{2}, m_2 - \frac{1}{2}, m_3))$$

$$u_2((y_1, y_2, y_3; x_1, x_2, x_3)) = m_2 < 1 + m_2 - \frac{1}{2} = u_2((0,1,0; \frac{1}{2}, m_2 - \frac{1}{2}, m_3))$$

and $((0,1,0; \frac{1}{2}, m_2 - \frac{1}{2}, m_3)) \in A(\{1,2\})$

$\Rightarrow \{1,2\}$ can impose upon $(y_1, y_2, y_3; x_1, x_2, x_3)$

$\downarrow$

Impossible

$\Rightarrow$ Candidates for actions in Core : $(0,1,0;1,m_2-1,m_3)$, $(0,0,1;1,m_2,m_3-1)$ 67.3
, that is,

Core $\leq \{(0,1,0;1,m_2-1,m_3), (0,0,1;1,m_2,m_3-1)\}$

- $(0,1,0;1,m_2-1,m_3) \in$ Core

$u_1((0,1,0;1,m_2-1,m_3)) = 1 \geq 0 = u_1((1,0,0;0,m_2,m_3))$

$u_2((0,1,0;1,m_2-1,m_3)) = 1+m_2-1 = m_2 \geq m_2 = u_2((1,0,0;0,m_2,m_3))$

$u_3((0,1,0;1,m_2-1,m_3)) = m_3 \geq m_3 = u_3((1,0,0;0,m_2,m_3))$

and $A(\{1\}) = A(\{2\}) = A(\{3\}) = \{(1,0,0;0,m_2,m_3)\}$

$\Rightarrow$ $\{1\}, \{2\}, \{3\}$ cannot improve upon $(0,1,0;1,m_2-1,m_3)$

If $(y_1,y_2,0;x_1,x_2,m_3) \in A(\{1,2\})$ is s.t.

$u_1(y_1,y_2,0;x_1,x_2,m_3) = x_1 > 1 = u_1(0,1,0;1,m_2-1,m_3)$,

then

$$\underbrace{u_2(y_1,y_2,0;x_1,x_2,m_3)}_{\substack{\shortparallel \\ y_2+x_2}} \leq \underset{\substack{\vee \\ y_2}}{1} + \underbrace{m_2-x_1}_{\substack{\shortparallel \\ x_2}} \leq m_2 = u_2(0,1,0;1,m_2-1,m_3)$$

$\Rightarrow$ $\{1,2\}$ cannot improve upon $(0,1,0;1,m_2-1,m_3)$

If $(y_1,0,y_3;x_1,m_2,x_3) \in A(\{1,3\})$ is s.t.

$u_1(y_1,0,y_3;x_1,m_2,x_3) = x_1 > 1 = u_1(0,1,0;1,m_2-1,m_3)$

then

$$\underbrace{u_3(y_1,0,y_3;x_1,m_2,x_3)}_{\substack{\shortparallel \\ y_3+x_3}} \leq \underset{\substack{\vee \\ y_3}}{1} + \underbrace{m_3-x_1}_{\substack{\shortparallel \\ x_3}} \leq m_3 = u_3(0,1,0;1,m_2-1,m_3)$$

$\Rightarrow$ $\{1,3\}$ cannot improve upon $(0,1,0;1,m_2-1,m_3)$

If $(1,0,0;0,x_2,x_3) \in A(\{2,3\})$ is s.t.

$u_2(1,0,0;0,x_2,x_3) = x_2 > m_2 = u_2(0,1,0;1,m_2-1,m_3)$

then

$u_3(1,0,0;0,x_2,x_3) = x_3 = m_3 + m_2 - x_2 \leq m_3 = u_3(0,1,0;1,m_2-1,m_3)$

$\Rightarrow$ $\{2,3\}$ cannot improve upon $(0,1,0;1,m_2-1,m_3)$

Suppose there exists $(y_1,y_2,y_3;x_1,x_2,x_3)\in A(\{1,2,3\})$ s.t.

$u_1(y_1,y_2,y_3;x_1,x_2,x_3)=x_1>1=u_1(0,1,0;1,m_2-1,m_3) \Rightarrow x_2+x_3=m_2+m_3-x_1<m_2+m_3-1$

$\Downarrow$

$x_2<m_2$ or $x_3<m_3$

$=u_2(\underline{\qquad\qquad})=y_2+x_2>m_2=u_2(\underline{\qquad\qquad})$

$u_3(\underline{\qquad\qquad})=y_3+x_3>m_3=u_3(\underline{\qquad\qquad})$

Assume $x_2<m_2$ ($x_3<m_3$ is similar)

$\Rightarrow y_2=1$

$y_3=0$

$\downarrow$

$x_3>m_3$

$\Rightarrow x_2=m_2+m_3-x_1-x_3<m_2+m_3-1-x_3$

$<m_2-1$

$\Rightarrow y_2+x_2=1+x_2<1+m_2-1=m_2$

$\downarrow$

Impossible

$\Rightarrow \{1,2,3\}$ cannot impose upon $(0,1,0;1,m_2-1,m_3)$

Similarly:

- $(0,0,1;1,m_2,m_3-1)\in$ Core

$\Rightarrow$ Core $=\{(0,1,0;1,m_2-1,m_3),\ (0,0,1;1,m_2,m_3-1)\}$

8.3.2 Small landowners

$$v(C) = \frac{\# C}{n} f(n)$$

$$\text{Core} = \left\{ \left(\frac{f(m)}{m}, \dots, \frac{f(n)}{n} \right) \right\}$$

8.3.3. Collective ownership

$$v(C) = f(n), \text{ if } \# C > \frac{n}{2}$$

$$= 0 \quad , \text{ if } \# C \le \frac{n}{2}$$

$$\text{Core} = \phi$$

.4 Illustration : exchanging homogeneous houses

House trading game with valuations $\overbrace{\beta_1 > \dots > \beta_n} \rightarrow$ nonowners

layer i has (large) wealth m_i

$\underbrace{\sigma_1 < \dots < \sigma_m} \rightarrow$ owners

$= \{\overbrace{1, \dots, n}^{\text{nonowners}}, \underbrace{n+1, \dots, n+m}_{\text{owners}}\}$

$\beta_1 > \sigma_1$, $\beta_n < \sigma_m$

and

$\beta_i \neq \sigma_j$. for all $1 \le i \le n$, $1 \le j \le m$

$(C) = \{ (y_1, \dots, y_{n+m}; x_1, \dots, x_{n+m}) \mid$

$(y_1, \dots, y_{n+m})$ is a C-allocation of the houses owned by members of C

$(x_1, \dots, x_{n+m})$ is a C-allocation of the money owned by members of C $\}$

$$_i(y_1, \dots, y_{n+m}; x_1, \dots, x_{n+m}) = y_i + \underbrace{(x_i - m_i)}_{\overset{\shortparallel}{r_i}}$$

↪ increase in the amount of money owned.

et $a_N = (y_1, \dots, y_{n+m}; x_1, \dots, x_{n+m})$ be an element of Core

$$\rho_j = m_j - x_j, \quad 1 \le j \le n$$

$$r_i = x_i - m_i; \quad n+1 \le i \le n+m$$

$$L^t = \{ i \mid n+1 \le i \le n+m \text{ and } y_i = 0 \}$$

= {owners with no house in a_N}

$$B^t = \{ j \mid 1 \le j \le n \text{ and } y_j = 1 \}$$

= {nonowners with a house in a_N}

$\Rightarrow \#L^t = \#B^t$

There exists k^t s.t. $\#L^t = \#B^t = k^t$

- $k^t > 1$

Otherwise $\{1, n+1\}$ can improve upon a_N

- If $j \notin B^t \Rightarrow \rho_j \le 0$
  otherwise $\{j\}$ can improve upon a_N
- If $j \notin B^t \Rightarrow \rho_j \geq 0$
  otherwise $N \setminus \{j\}$ can improve upon a_N

$\Rightarrow$ • If $j \notin B^t \Rightarrow \rho_j = 0$

- If $i \notin L^t \Rightarrow r_i \geq 0$
  otherwise $\{i\}$ can improve upon a_N
- If $i \notin L^t \Rightarrow r_i \le 0$
  otherwise $N \setminus \{i\}$ can improve upon a_N

$\Rightarrow$ • If $i \notin L^t \Rightarrow r_i = 0$

- If $j \in B^t, i \in L^t \Rightarrow r_i \geq \rho_j$
  Otherwise $\{i, j\}$ can improve upon a_N

← $\sum_{i \in L^t} r_i = \sum_{j \in B^t} \rho_j$ and $\#L^t = \#B^t = k^t$

→ • If $j \in B^t, i \in L^t \Rightarrow r_i = \rho_j$

$\Rightarrow$ There exists p^t s.t. $r_i = \rho_j = p^t$, for all $i \in L^t, j \in B^t$

- If $n+1 \leq i \leq n+m$ and $\sigma_{i-m} < \rho^* \Rightarrow i \in L^*$

  Otherwise, for all j in B^*,

  $\{i, j\}$ can improve upon a_N

- If $n+1 \leq i \leq n+m$ and $\sigma_{i-m} > \rho^b \Rightarrow i \notin L^*$

  Otherwise $\{i\}$ can improve upon a_N

$\Rightarrow$ • $L^* = \{n+1, \ldots, n+k^*\}$

and $\sigma_{k^*} \leq \rho^* \leq \sigma_{k^*+1}$

buyer j's valuation

- If $1 \leq j \leq n$ and $\beta_j > \rho^* \Rightarrow j \in B^*$

  Otherwise, for all $i \in L^*$,

  $\{i, j\}$ can improve upon a_N

- If $1 \leq j \leq n$ and $\beta_j < \rho^* \Rightarrow j \notin B^*$

  Otherwise, $\{j\}$ can improve upon a_N

$\Rightarrow$ • $B^* = \{1, \ldots, k^*\}$

and $\beta^*_{k+1} \leq \rho^* \leq \beta^*_{k^*}$

$$\Downarrow$$

$$k^* = \max \{k \mid \beta_k > \sigma_k\}$$

$$\rho^* \in [\max(\sigma_{k^*}, \beta_{k^*+1}), \min(\beta_{k^*}, \sigma_{k^*+1})]$$

Exercise 259.1

$\Rightarrow$ Core $\subseteq$

$$E = \{(\underset{1}{1}, \ldots, \underset{k^*}{1}, \underset{k^*+1}{0}, \ldots, \underset{n}{0}, \underset{n+1}{0}, \ldots, \underset{n+k^*}{0}, \underset{n+k^*+1}{1}, \ldots, \underset{n+m}{1}; m_1 - \rho^*, \ldots, m_{k^*} - \rho^*, m_{k^*+1}, \ldots, m_n, m_{n+1} + \rho^*, \ldots, m_{n+k^*} + \rho^*, m_{n+k^*+1}, \ldots, m_{n+m}) \quad \rho^* \in [\max(\sigma_{k^*}, \beta_{k^*+1}), \min(\beta_{k^*}, \sigma_{k^*+1})]\}$$

with $k^+ = \max \{k \mid \beta_k > \sigma_k\}$

Let $a_N \in E$

$\hookrightarrow$ obtained from ρ^* in $[\max(\sigma_k, \beta_{k+1}), \min(\beta_k, \sigma_{k+1})]$

<u>then</u>: $u_j(a_N) = \max(\beta_j, \rho^*) - \rho^*$, for all $1 \leq j \leq n$

$u_i(a_N) = \max(\sigma_{i-n}, \rho^*)$, for all $n+1 \leq i \leq n+m$

<u>Notation</u>: $\left[\begin{array}{l} V_j = \beta_j, \text{ for all } 1 \leq j \leq n \\ V_i = \sigma_{i-n}, \text{ for all } n+1 \leq i \leq n+m \end{array}\right.$ (V_i is player i's valuation)

Suppose there exists C s.t. C can improve upon a_N

$\# C \cap \{1, \dots, n\} = b$ (# of nonowners in C)

$\# C \cap \{n+1, \dots, n+m\} = l$ (# of owners in C)

Let C^* be the subset of C s.t.

$\# C^* = l$
and
$V_i \geq V_{i'}$, for all $i \in C^*$, $i' \in C \setminus C^*$

(C^* is the set of the l members of C with the highest valuations)

$\Downarrow$

there exists a in $A(C)$ s.t.
$u_i(a) > u_i(a_N)$, for all $i \in C$

$\Downarrow$

$$\sum_{i \in C^*} V_i \geq \sum_{i \in C} u_i(a) > \sum_{i \in C} u_i(a_N) = \sum_{i \in C} \max(V_i, \rho^*) - b\rho^*$$

$\Downarrow$

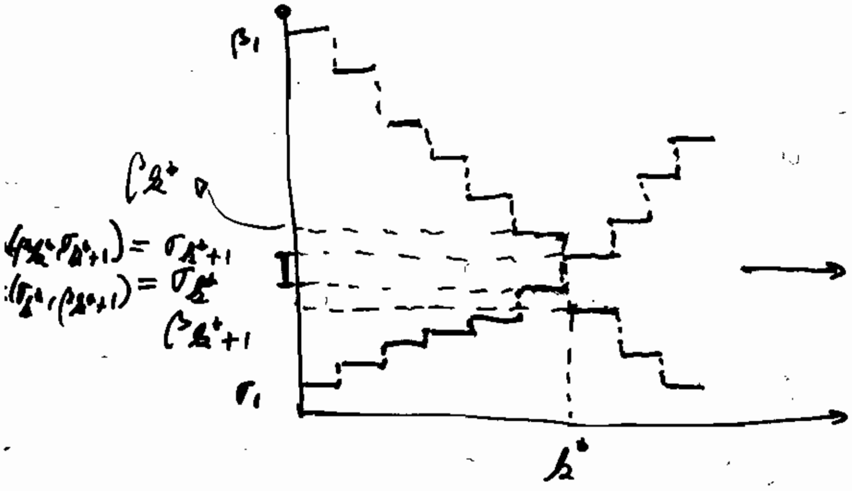
$$\sum_{i \in C^*} V_i > \sum_{i \in C^*} \max(V_i, \rho^*) + \underbrace{\sum_{i \in C \setminus C^*} \max(V_i, \rho^*) - b\rho^*}_{\#(C \setminus C^*) = b + l - l = b} \geq \sum_{i \in C^*} \max(V_i, \rho^*)$$

$\hookrightarrow$ Impossible

$\Rightarrow a_N \in$ Core

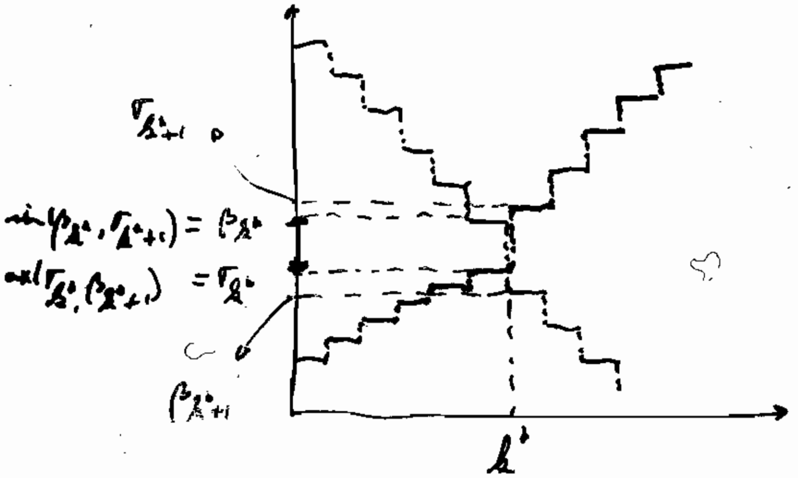
$\Rightarrow$ Core $= E$

Examples



Rem: For all actions in the core, exactly k^* horses are exchanged

- Owner k^* always trades (even if $p^* = \sigma_{k^*}$)
- Owner k^*+1 never trades (even if $p^* = \sigma_{k^*+1}$)



Exercise 256.1

Exercise 256.2